Stat 605: Problem set 6

1. Exercise 10.6, p. 73

2. Exercise 10.7, P. 73

3. The Borel-Cantelli Lemma states that for a sequence of independent A_n we have $P(\limsup_n A_n) = 0$ if and only if $\sum_n P(A_n) < \infty$. Prove the alternative version that we have $P(\limsup_n A_n) = 1$ if and only if $\sum_n P(A_n) = \infty$.
   Hint: You will need to revisit the proof we have done in class.

   (a) Suppose X_n are independent random variable $P(X_n = 1) = p_n = 1 - P(X_n = 0)$. Show that $P(\lim_{n\to\infty} X_n = 0) = 1$ if and only if $\sum_n p_n < \infty$.

   (b) Suppose X_n is a sequence of independent random variables with $P(X_n = 1) = p = 1 - P(X_n = 0)$. What is the probability that the pattern $1, 0, 1$ occurs infinitely often. Hint: Let $A_k = (X_k = 1, X_{k+1} = 0, X_{k+2} = 1)$ and consider $A_1, A_4, A_7, \cdots$.

4. Show that if X_n is a sequence of independent random variables and $S_n = X_1 + \cdots X_n$ the random variables $\limsup_{n\to\infty} \frac{1}{n} S_n$ and $\liminf_{n\to\infty} \frac{1}{n} S_n$ are constant almost surely.

5. Let c_n is a sequence of independent random variables and consider the power series $\sum_{n=1}^{\infty} c_n x^n$. Show that the radius of convergence of the power series in constant.
   Hint: The convergence radius is $\rho = \limsup |c_n|^{1/n}$.

6. Consider the probability space $[0, 1]$ with the Borel σ-algebra and the uniform measure P on $[0, 1]$. Consider the random variable $X(\omega) = \omega$. Is there a bounded random variable which is both independent of X and not constant P almost surely?

7. Suppose $X \geq 0$ is a nonnegative random variable on the probability space $(\Omega, \mathcal{A}, P)$. Show that

$$E[X] = \int_0^\infty P(X > s)ds \,.$$

   To do this consider the product measure $P \otimes m$ of P with the Lebesgue measure on $[0, \infty)$ and use Fubini-Tonelli Theorem. Note that

$$E[X] = \int X(\omega) dP(w) = P \times m(\{(\omega, s)\,; 0 \leq s < X(\omega)\})$$

8. Exercise 11.2, p.84

9. Exercise 11.3, p. 84