Stat 605: Problem set 4

1. **The limsup and liminf of sequence of numbers.** Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers. A number b is an accumulation point of the sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = b$.
Consider the sets

$$X = \{x\,;\ \text{infinitely many } x_n \text{ are } > x\}\,,\quad Y = \{x\,;\ \text{infinitely many } x_n \text{ are } < x\}\,.$$

and define

$$\xi := \sup X\,,\quad \eta := \inf Y\,.$$

(a) Prove that ξ is the largest accumulation point of $\{x_n\}$ and that η is the smallest accumulation point of $\{x_n\}$. We then write

$$\xi = \limsup_{n\to\infty} x_n \qquad \text{the limit superior of the sequence } \{x_n\}\,.$$

$$\eta = \liminf_{n\to\infty} x_n \qquad \text{the limit inferior of the sequence } \{x_n\}\,.$$

(b) Show the formulas

$$\limsup_{n\to\infty} x_n \;=\; \lim_{n\to\infty}\sup_{k\geq n} x_k \;=\; \inf_{n\geq 1}\sup_{k\geq n} x_k\,.$$

$$\liminf_{n\to\infty} x_n \;=\; \lim_{n\to\infty}\inf_{k\geq n} x_k \;=\; \sup_{n\geq 1}\inf_{k\geq n} x_k\,.$$

2. **The limsup and liminf of sequence of sets.** Let $\mathcal{A}$ be a σ-algebra of subsets of Ω and let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of elements of $\mathcal{A}$. We define

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \left\{x\in\mathbf{R}^d\,;\, x\in A_n \text{ for infinitely many } n\right\}\\ \liminf_{n\to\infty} A_n &= \left\{x\in\mathbf{R}^d\,;\, x\in A_n \text{ for all but finitely many } n\right\}\,.\end{aligned}$$

(a) Show that

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\,,\qquad \text{and } \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\,.$$

(b) Show that

$$P\left(\liminf_{n\to\infty} E_n\right) \leq \liminf_{n\to\infty} P\left(E_n\right) \leq \limsup_{n\to\infty} P\left(E_n\right) \leq P\left(\limsup_{n\to\infty} E_n\right) \qquad (1)$$

3. **(Limits of sets)** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\{A_n\}_{n=1}^{\infty}$ be a collection of elements of $\mathcal{A}$. We say that A_n converges to A and write $\lim_{n\to\infty} A_n = A$ if the indicator function $\mathbf{1}_{A_n}(\omega)$ converges to $\mathbf{1}_A(\omega)$ for all $\omega \in \Omega$.

   (a) Show that if A_n converges to A then

   $$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n \, .$$

   (b) Show that if $\lim_{n\to\infty} A_n = A$ then $\lim_{n\to\infty} P(A_n) = P(A)$. *Hint:* Use (a).

4. Suppose X_n is a sequence of random variables such that X_n converges to X almost surely. Show that if f is a continuous function then $f(X_n)$ converges almost surely to $f(X)$.

5. Exercises 7.11, 7.12, 7.13, p. 45.

6. Exercises 9.5 and 9.6 p. 61 *Hint:* Use dominated convergence theorem to prove countable additivity of Q.