Evolutionary game theory, Problem set 2

1. Show that a game $(\Gamma, S, (\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma})$ is a potential game (i.e., $\boldsymbol{\pi}_\gamma = \boldsymbol{\pi}$ for all $\gamma \in \Gamma$) if and only if for all $i \in S_\gamma$

$$\frac{\partial}{\partial x_{i,\gamma}} \boldsymbol{\pi}_\xi(\mathbf{e}_{j,\xi}, \mathbf{p}_{-\xi}) \;=\; \frac{\partial}{\partial x_{j,\xi}} \boldsymbol{\pi}_\gamma(\mathbf{e}_{i,\gamma}, \mathbf{p}_{-\gamma})$$

   for all $\gamma \in \Gamma$, $\xi \in \Gamma$, $i \in S_\gamma$, $j \in S_\xi$.

2. Show that a game $(\Gamma, S, (\boldsymbol{\pi}_\gamma)_{\gamma\in\Gamma})$ is Nash equivalent to a potential game if and only if there exists a $\boldsymbol{\pi}$ such that for all $\gamma \in \Gamma$ we have

$$\boldsymbol{\pi}_\gamma(\mathbf{p}_\gamma, \mathbf{p}_{-\gamma}) - \boldsymbol{\pi}_\gamma(\mathbf{q}_\gamma, \mathbf{p}_{-\gamma}) \;=\; \boldsymbol{\pi}(\mathbf{p}_\gamma, \mathbf{p}_{-\gamma}) - \boldsymbol{\pi}(\mathbf{q}_\gamma, \mathbf{p}_{-\gamma}).$$

   for all $\gamma \in \Gamma$

3. Consider a two players zero-sum game with players α and β and strategies $S_\alpha = \{s_1, \cdots, s_n\}$ and $S_\beta = \{t_1, \cdots, t_m\}$. Suppose that s_i, t_k and s_j, t_l are Nash equilibria for the game with $i \neq j$ and $k \neq l$. Show that s_i, t_l and s_j, t_k are also NE.

4. Suppose the two player games with players $\Gamma = \{\alpha, \beta\}$ is a symmetric game and a zero-sum game. Show that the value of the game is always 0.

5. Consider the following zeros-sum game called Morra. Two players simultaneously show either one or two fingers and at the same time guesses the number of fingers shown by the other player. If both guess correctly or none does then the payoff is zero to both. If one guesses and the other does not, the payoff to the guesser is the sum of the fingers shown. For each of the players there are four strategies, let us call them $(1,1), (1,2), (2,1), (2,2)$ where (i,j) means show i fingers and guess j fingers.

   (a) Write down the game matrix

   (b) Find the value of the game and solve it using the minimax theorem. (Use the previous problem)

6. Find the NE and the ESS for the symmetric game with payoff matrix

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 10 & 1 \\ 10 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

7. A closed nonempty subset E of Δ is said to be an *evolutionary stable set* if for each $\mathbf{p} \in E$ there exists a neighborhood W of $\mathbf{p}$ such that

$$\boldsymbol{\pi}(\mathbf{p}, \mathbf{q}) \geq \boldsymbol{\pi}(\mathbf{q}, \mathbf{q}) \qquad (1)$$

with strict inequality if $\mathbf{q} \notin E$. Show that if $\mathbf{p} \in E$ then $\mathbf{p}$ is a NE. Show that if the singleton $\{\mathbf{p}\}$ is a evolutionary stable set if and only if $\mathbf{p}$ is an ESS. Compute the NE and the evolutionary stable sets for the game with payoff matrix

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} .$$

8. Consider the Hawk-Dove game with payoff matrix

$$\boldsymbol{\pi} = \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix}$$

and NE (and ESS) $\mathbf{p} = (2/3, 1/3)$. Imagine that the population is invaded with *two* populations of mutants, half of them using the pure strategy $\mathbf{e}_1$ and the other half using the pure strategy $\mathbf{e}_2$. The resulting mutant population is described by the mixed strategy $\mathbf{r} = \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$ and the new population is $\mathbf{q} = \epsilon\mathbf{r} + (1-\epsilon)\mathbf{p}$. Show that

$$\boldsymbol{\pi}(\mathbf{e}_1, \mathbf{q}) > \boldsymbol{\pi}(\mathbf{p}, \mathbf{q})$$

that is the mutant of type 1 have a better payoff than the incumbent population. Does this contradict ESS? Explain.