Math 697: Homework 3

**Exercise 1** A mouse is performing a symmetric random walk on the positive integer {0, 1, 2, 3, ...}; if it is in state $i$ it is equally likely to move to state $i - 1$ or $i + 1$. The state 0 is the mouse’s home filled with lots of tasty cheese. If the mouse ever reaches its home it will stay there forever. On the other hand there is a bad cat who tries to catch the mouse and each time the mouse moves there is a probability $1/5$ that the cat will kill the mouse.

To describe this process as a Markov chain consider an extra state $*$ which corresponds to the mouse being dead. The state space is $S = \{*, 0, 1, 2, \cdots\}$ and $X_n$ denotes the position of the mouse at time $n$. Compute the corresponding transition matrix.

Compute the probability that the mouse reaches safety if it starts in state $i$, i.e.,

$$p_i \equiv P\{X_n = 0 \text{ for some } n \mid X_0 = i\}, i = 0, 1, 2, \cdots$$

**Exercise 2** Problem 2.1, p. 57

**Exercise 3** Problem 2.2, p. 57

**Exercise 4** Problem 2.4, p. 58

**Exercise 5** Problem 2.6, p. 58

**Exercise 6** Problem 2.7, p. 58

**Exercise 7**
Consider the following Markov chain. At times $n = 1, 2, 3, \cdots$ $\xi_n$ particles are added in a box where $\xi_n$ are i.i.d. random variables with a Poisson distribution with parameter $\lambda$, i.e.,

$$P\{\xi_n = k\} = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Suppose that any of the particles in the box at time $n$ independently of all other particles and of how the particles are added to the box has probability $p$ of remaining in the box at time $n + 1$ and probability $q = 1 - p$ of being removed from the box. The number of particles in the box at time $n$, $X_n$ is a Markov chain which can be expressed as

$$X_{n+1} = \xi_{n+1} + R(X_n),$$

where $R(X_n)$ denotes the number of particles present at time $n$ and which remain at time $n + 1$.

1. Show that the Markov chain is irreducible and aperiodic.

2. As a preparation prove the following fact. Let $Z$ be a Poisson random variable with parameter $\mu$ describing the number of some items. The items occur in two types, type $A$ with probability $p_A$ and type $B$ with probability $p_B = 1 - p_A$ and Let $Z = Z_A + Z_B$ where $Z_A$ are the number of items of type $A$ and $Z_B$ are the number of items of type $B$. Show that $Z_A$ and $Z_B$ are Poisson random variables with parameter $\mu p_A$ and $\mu p_B$. 

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3. Use $A$ to show that if the initial distribution of $X_0$ is Poisson with parameter $\nu$ then $X_1$ has also a Poisson distribution. Compute the probability distribution of $X_n$ and determine the limiting and stationary distribution.

**Exercise 8** Problem 2.8, p. 58

**Exercise 9** Consider a branching process with offspring distribution given by $p_n$. One makes this process irreducible by asserting that if the population ever dies out, then in the next generation one new individual appears (i.e. $P_{01} = 1$). Determine for which values of $p_n$ the chain is positive recurrent, null recurrent, transient.

**Exercise 10** An electric light that has survived $n$ seconds fails during the $(n + 1)$th second with probability $q$ (with $0 < q < 1$).

1. Let $X_n = 1$ if the light is functioning at time $n$ seconds, and $X_n = 0$ otherwise. Let $T$ be the time of failure of the light (in seconds), i.e.,

$$T = \inf\{n; X_n = 0\}.$$  

Determine $E[T]$.

2. A building contains $m$ lights of the type described above, which behave independently of each other. At time 0 they are all functioning. Let $Y_n$ denote the number of lights functioning at time $n$. Specify the transition matrix of $Y_n$.

3. Find the moment generating function

$$\phi_n(s) = E[s^{Y_n}]$$  

of $Y_n$. *Hint: Express $\phi_n$ in terms of $\phi_{n-1}$ and solve the recursion relation.*

4. Use the moment generating function to find $P\{Y_n = 0\}$ and $E[Y_n]$.

**Exercise 11** Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability $p_j$ that $j$ more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break.