Math 697: Midterm

Problem 1 **(a)** Suppose X_n is a finite state Markov chain. Show that the set of stationary distributions for X_n is a convex subset of the set of all probability vectors.
Hint: Recall that a subset A of a vector space is convex if $x \in A$ and $y \in A$ implies that $\alpha x + (1-\alpha)y \in A$ for all $0 \le \alpha \le 1$.

(b) Consider the Markov chain on the state space $\{1, 2, \cdots, 7\}$ with transition matrix

$$P = \begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

Find all communication classes. Determine if they are closed or transient, periodic or aperiodic. Compute all stationary distributions.

(c) For the Markov chain of **(b)** compute $\lim_{n\to\infty} P^n(6,i)$ and $\lim_{n\to\infty} P^n(5,i)$ for all $i \in S$.

Problem 2 **(a)** Suppose X_n is a Markov chain with state space S and $f : S \to T$ is a map from S to some set T. Show that $Y_n = f(X_n)$ is a Markov chain on the state space $S' = \{f(j)\,;\, j \in S\}$ if f is one-to-one. What if f is not one-to-one?
(b) Suppose X_n is an irreducible Markov chain on the finite state space S with stationary distribution π. Let $Y_n \equiv (X_n, X_{n+1})$. Show that Y_n is a Markov chain. What is the state space for Y_n? What are the transition probabilities? What is the stationary distribution?
(c) Suppose X_n and Y_n are two independent irreducible Markov chains with finite state space S and T respectively, transition probabilities P and Q respectively, and stationary distribution π and ν respectively. Show that $Z_n = (X_n, Y_n)$ is a Markov chain, find the transition probabilities, and the stationary distribution.
(d) A neighborhood has 2 bars, called 1 and 2. B.J. visits one of two bars every night, starting in bar 1 according to the Markov chain with transition matrix

$$P = \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix}$$

while C.J. visits one of two same bars every night, starting in bar 2 according to the Markov chain with transition matrix

$$P = \begin{pmatrix} .3 & .7 \\ .7 & .3 \end{pmatrix}$$

Find the expected time until they are in the same bar and the probability they meet in bar 2.
Hint: Use **(c)**.

Problem 3 Consider a sequence of random numbers $U_1, U_2, U_3, \cdots$ (i.e. uniform random variables on $[0,1]$). Let N be the first one that is greater than its immediate predecessor, i.e.,

$$N \;=\; \min\{n\,;\, n \geq 2\,, U_n > U_{n-1}\}$$

We will also need later the random variable

$$M = \min\{n\,;\, n \geq 2\,, 1-U_n > 1-U_{n-1}\} \;=\; \min\{n\,;\, n \geq 2\,, U_n < U_{n-1}\}$$

which has the same distribution as N.

(a) Show that $P\{N > n\} = \frac{1}{n!}$ and deduce from this that $E[N] = e \left(= \sum_{n=0}^{\infty} \frac{1}{n!}\right)$.

(b) Using the results of **(a)** compute the variance of the Monte-Carlo algorithm

$$I_L \;=\; \frac{1}{L}\sum_{j=1}^{L} N_j$$

which is an estimator for the number e.

(c) Observe that with probability $1/2$ (i.e., if $U_1 > U_2$), $N = 2$, and that with probability $1/2$, $N = 2 + K$ where where K is the number of additional random numbers necessary to be observed until one is observed to be greater than its prededessor, given that $U_2 < U_1$. Formally let K be the random variable which takes the value $1, 2, 3, \cdots$ and whose p.d.f is given by

$$P\{K = j\} \;=\; P\{N = 2 + j \,|\, U_1 > U_2\}$$

Use this relation to compute $E[K]$ and $\mathrm{Var}(K)$.

(d) If one uses one sequence of random numbers to generate both N and M then N and M are not independent (for example one of them is always 2). Use (**c**) to compute in this case $\mathrm{Var}(N + M)$.

(e) Formulate a Monte-Carlo algorithm to estimate e using M and N and shows that it has smaller variance than the one in **(b)**.