Math 697: Homework 3

Exercise 1 Let P be a transition probability matrix and assume that there exists a stationary distribution $\pi = (\pi(1), \cdots, \pi(N))$ with $\pi(j) > 0$ for all j. We now define a new matrix, $\bar{P}$, called the *time reversed* transition probabilities, by

$$\bar{P}ij \;=\; \frac{\pi(j)}{\pi(i)} P_{ji} \,.$$

(a) Show that $\bar{P}_{ij}$ is a stochastic matrix and that π is again a stationary distribution for $\bar{P}$.

(b) Let X_n be the Markov chain with transition matrix P and initial distribution π. Let $\bar{X}_n$ be the Markov chain with transition matrix $\bar{P}$ and initial distribution π. Show that

$$P\left\{\bar{X}_0 = i_0, \bar{X}_1 = i_1, \cdots, \bar{X}_n = i_n\right\} \;=\; P\left\{X_n = i_0, X_{n-1} = i_1, \cdots X_0 = i_n\right\} \,.$$

The Markov chain $\bar{X}_n$ is called the *time reversed* chain for the Markov chain X_n.

Exercise 2 Suppose X_n an irreducible Markov chain with stationary distribution π. Furthermore we assume that for all pair i, j we have $P(i, j) > 0$ iff $P(j, i) > 0$. Show that the following are equivalent

(i) The Markov chain satisfies detailed balance, i.e., for all i, j we have

$$\pi(i)P(i, j) = \pi(j)P(j, i)\,.$$

(ii) For any n and any sequence of state $i, i_1, i_2, \cdots i_n$ we have

$$P(i, i_1)P(i_1, i_2) \cdots P(i_{n-1}, i_n)P(i_n, i) \;=\; P(i, i_n)P(i_n, i_{n-1}) \cdots P(i_2, i_1)P(i_1, i)$$

Hint: To show $(ii) \Rightarrow (i)$ use the convergence to stationary distribution.

Exercise 3 (Metropolis-Hastings algorithm) Let $\pi(i) > 0$ be a probability distribution on the finite state space S. Let $Q(i, j)$ be a transition probability matrix (not necessarily symmetric). Set

$$T(i, j) \;=\; \frac{\pi(j)Q(j, i)}{\pi(i)Q(i, j)}$$

and suppose $A : [0, \infty] \to [0, 1]$ be a function such that $A(z) = zA(1/z)$ for all $z \in [0, \infty]$. Finally we define for $i \neq j$

$$P(i, j) \;=\; Q(i, j)A(T(i, j))$$

and $P(i, i) = 1 - \sum_{j \neq i} P(i, j)$. You can think of $A(T(i, j))$ has the acceptance probability for the proposed transition from i to j.

(a) Suppose $Q(i, j) > 0$ for all i, j. Prove that P is the transition matrix of a reversible Markov chain which satisfies detailed balance and has stationary distribution π. Prove also that P is irreducible.

(b) Suppose now that $Q(i, j)$ might be 0 for some values of i, j. Prove that P is still well-defined and satisfies detailed balance but that P might not be irreducible even if Q is irreducible.

(c) Find the values of a and b for which $A(z) = \frac{z^a}{1+z^b}$ can be used.

(d) Show that $A(z) = \min\{1, z\}$ can be used and that it leads to the Metropolis algorithm when Q is symmetric. In which sense is that A the "best choice of A?

Exercise 4 Consider the Markov chain on $S = \{0, 1, 2, 3, \cdots\}$ with transition probabilities

$$P(0,0) = 1 - p_0, \quad P(0,1) = p_0$$

$$P(j, j-1) = 1 - p_j, \quad P(j, j+1) = p_j.$$

(a) Under which conditions on p_j is the Markov chain positive recurrent. *Hint:* Use detailed balance.

(b) Suppose that you are given a probability distribution $\pi(j)$ on S. Under which conditions can you choose $p(0), p(1), \cdots$ such that the stationary distribution of the Markov chain in **(a)** is π?

(c) Find b_i such that π is a Poisson distribution with parameter λ.

(d) Choose $p_0 = 1/2$ and $p_i = 1/2$ for $i \geq 1$ and use now Metropolis-Hastings to generate a Poisson distribution with parameter λ. Compare with **(c)**.

Exercise 5 Discrete queueing model with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

$$P(0,0) = 1 - p, \quad P(0,1) = p$$

$$P(j, j-1) = q(1-p), \quad P(j,j) = pq + (1-p)(1-q), \quad P(j, j+1) = p(1-q).$$

Determine when the Markov chain is transient, recurrent, or positive recurrent. In the positive recurrent case compute the stationary distribution and the length of the queue in equilibrium. In the transient case compute the probability $\alpha(j)$ of ever reaching 0 starting from i.

Exercise 6 Let $p(k)$, $k = 0, 1, 2, 3, \cdots$. Consider the Markov chain on $S = \{0, 1, 2, 3, cdots\}$ with transition probabilities

$$P(0, k) = p(k), \quad k = 0, 1, 2, 3, \cdots$$

$$P(k, k-1) = 1, \quad k \geq 1.$$

Under which conditions on $p(k)$ is the Markov chain positive recurrent? In that case compute the stationary distribution π.

Exercise 7 A mouse is performing a symmetric random walk on the positive integer $\{0, 1, 2, 3, \cdots\}$: if it is in state i it is equally likely to move to state $i-1$ or $i+1$. The state 0 is the mouse's home filled with lots of tasty cheese. If the mouse ever reaches its home it will stay there forever. On the other hand there is a bad cat who tries to catch the mouse and each time the mouse moves there is a probability $1/5$ that the cat will kill the mouse.

To describe this process as a Markov chain consider an extra state $*$ which corresponds to the mouse being dead. The state space is $S = \{*, 0, 1, 2, \cdots\}$ and X_n denotes the position of the mouse at time n. Compute the corresponding transition matrix.

Compute the probability that the mouse reaches safety if it starts in state i, i.e.,

$$p_i \equiv P\{X_n = 0 \text{ for some n} \mid X_0 = i\}, i = 0, 1, 2, \cdots$$

Exercise 8 Given a branching process with the following offspring distributions determine the extinction probability a.

(a) $p(0) = .25, p(1) = .4, p(2) = .35$

(b) $p(0) = 5, p(1) = .1, p(3) = .4$

(c) $p(0) = .62, p(1) = .30, p(2) = .02, p(3) = .02, p(6) = .02, p(13) = .02$

(c) $p(i) = (1 - q)q^i$

Exercise 9 Consider a branching process with offspring distribution given by p_n. One makes this process irreducible by asserting that if the the population ever dies out, then in the next generation one new individual appears (i.e. $P_{01} = 1$). Determine for which values of p_n the chain is positive recurrent, null recurrent, transient.

Exercise 10 Jamie is working in a bookstore, ordering books that are not in store and that the customers request. Each order takes 5 minutes to complete. While each order is being filled there is a probability p_j that j more customers arrive with $p_0 = .2$, $p_1 = .2$, $p_2 = .6$. Jamie cannot take a coffee break until a service is completed and no one is waiting in line to order a book. When Jamie starts her shift there is one customer waiting. What is the probability that she ever will take a coffee break.