Math 697: Homework 2

Exercise 1

The Smiths receive the paper every morning, they read it during breakfast, and place it on a pile after reading it. Each afternoon, with probability 1/3 someone takes all papers in the pile and put them in the recycling bin. Also if ever there at least five papers in the pile, Mr Smith, with probability one, take the papers to the bin in the afternoon.

1. Consider the number of papers in the pile in the evening and describe it with a Markov chain. What are the state space and transition probabilities?

2. Show that there is a unique limiting distribution. Compute it.

3. After a long time what would be the expected number of papers in the pile?

4. Assume that the piles starts with 0 papers. What is the expected time until the pile will have again 0 papers.

5. Estimate the number of times in a year that there are exactly two papers in the pile two days in a row.

Exercise 2 1. Consider a Markov chain with state space $\{0, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{pmatrix} .$$

What are the communication classes. Which ones are closed and which one are transient?

2. Consider a Markov chain with state $\{1, \cdots, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Is the chain irreducible? What is the period of the chain. Let $\tau^{(1)}$ be the first return time to state 1. Compute directly $E[\tau^{(1)}|X_0 = 1]$. Compute the stationary distribution π.

Exercise 3 The weather is in two states: rainy or not rainy and it depends on the weather in the previous two days. If it was rainy yesterday and and the day before yesterday it will be rainy today with probability 0.8. If it was not rainy yesterday and the day before yesterday it will be rainy today with probability 0.2. If it was rainy yesterday but not rainy the day before yesterday, then the weather will be rainy today with probability 0.5. If it was not rainy yesterday but rainy the day before yesterday, then the weather will be rainy today with probability 0.4.

1. Explain why it is not reasonable to model the weather on a single day as a Markov chain.
2. Show that one can construct a Markov chain by taking as a state the weather in two consecutive days. Write the corresponding transition probabilities (it is a 4×4 matrix).
3. Compute the probability that it is going to rain tomorrow given that it rained yesterday but not the day before yesterday

Exercise 4 Jane possesses r umbrellas which she uses going from her home to her office in the morning and vice versa in the evening. If it rains in the morning or in the evening she will take an umbrella with her provided there is one available. Assume that independent of the past it will rain in the morning or evening with probability p. Let X_n denote the number of umbrellas at her home before she gets to work.

1. Give the state space and the transition probabilities describing the Markov chain X_n.
2. Find the limiting probabilities π_j, $j = 0, 1, \ldots, r$.
3. In the long run, what fraction of the time does Jane get wet?

Exercise 5

Consider an irreducible Markov chain X_n with a finite state space. Let $\tau^{(j)}$ be the first return time to state j, i.e.,

$$\tau^{(j)} = \min\{n \geq 1, X_n = j\}.$$

Let

$$M_{ij} = E[\tau^{(j)} | X_0 = i]$$

denote the expected return time to the state j, starting from i.

1. Condition on on the position of chain at time 1 and show that

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}$$

2. Let π_i be the stationary distribution. Multiplying both sides by π_i and summing over i show that

$$\pi_j = \frac{1}{M_{jj}}.$$

Exercise 6 The Markov property means that the future depends on the present but not on the past, i.e.,

$$P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}, \cdots X_0 = i_0\} = P\{X_n = i_n \,|\, X_{n-1} = i_{n-1}\}.$$

1. Show that the Markov property implies that the past depends only on the present but not on the future, i.e.,

$$P\{X_0 = i_0 \,|\, X_1 = i_1, \cdots X_n = i_n\} = P\{X_0 = i_0 \,|\, X_1 = i_1\}.$$

2. Show that the Markov property also implies that, given the present, the past and the future are independent, i.e.,

$$P\{X_{n+1} = i_{n+1}\,,\, X_{n-1} = i_{n-1} \,|\, X_n = i_n\} \\ = P\{X_{n+1} = i_{n+1} \,|\, X_n = i_n\} P\{X_{n-1} = i_{n-1} \,|\, X_n = i_n\}\,.$$

Hint: Fix an arbitrary initial condition μ and use the formula for $P\{X_0 = i_0, \cdots X_n = i_n\}$

Exercise 7 Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers. Define the sequence $\{b_n\}_{n\geq 1}$ by

$$b_n = \frac{a_0 + \cdots + a_{n-1}}{n} = \frac{1}{n}\sum_{j=0}^{n-1} a_j\,.$$

Show that if $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} b_n = a$. The convergence of $\{b_n\}$ however does not imply the convergence of $\{a_n\}$.

Exercise 8

If the state space is even moderately large it can be quite difficult and quite tedious to compute the stationary distribution π for the Markov chain with transition probability P, i.e., to solve $\pi P = \pi$. The purpose of this exercise is to derive a simple algorithm to compute stationary distributions which reduces the problem to invert a matrix (and this is easy to do on your computer).

We define the following matrices: I is the $N \times N$ identity matrix and M is the $N \times N$ matrix whose entries are all 1, i.e.,

$$I = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

Algorithm for the stationary distribution: Suppose X_n is an irreducible Markov chain. Then the unique stationary distribution π is given by

$$\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}\,.$$

1. Assume first that the matrix $(I - P + M)$ is invertible. Show then π is given by by $\pi = (1, 1, \cdots, 1)\,(I - P + M)^{-1}$.

2. You need to prove now that $(I - P + M)$ is invertible. This is equivalent to prove that $(I - P + M)x = 0$ implies $x = 0$. To do this

   (a) Multiply $(I - P + M)x = 0$ by on the left by π and deduce from this that $Mx = 0$. Thus $Px = x$.

   (b) Use that the only solutions of $Px = x$ are of the form $x = c(1 \cdots, 1)^T$. Conclude.