

## Math 697: Final exam

This is an exam. Use your classnotes and text book, but nothing else.

**Problem 1** Potential customers arrive at a full-service gas station according to a Poisson process with a rate of 20 cars per hour. The gas station has one pump and can accommodate a maximum of three cars (including the one being currently attended at the pump). If there are three cars in the station the arriving cars do not enter the gas station. Suppose that the amount of time required to service a car is exponentially distributed with mean 5 minutes.

- (a) What fraction of the gas station attendant's time will be spent servicing cars?
- (b) What fraction of the potential customers are lost?
- (c) What is the average waiting time at the pump for the customers which actually enter the station?

**Problem 2 (a)** Consider a  $M/M/1$  queue where customers arrive at a service station according to a Poisson process at rate  $\lambda$ . The service time has an exponential distribution with parameter  $\mu$  and we assume  $\lambda < \mu$ . Show that if the system is in *its stationary state* then the customers *exit the service station* according to a Poisson process rate with rate  $\lambda$ .

(b) Consider a facility which consists of 2 service station labelled  $A$  and  $B$ . A customer first goes to service station  $A$  waits in line if the station is not empty, and then after completion of service goes to service station  $B$  and then waits in line to be served. Assume that the service at stations  $A$  and  $B$  are exponential distributed with parameter  $\mu_A$  and  $\mu_B$  respectively and that customers enter the facility according to a Poisson process with parameters  $\lambda$ . The state of the system is described by a vector  $\mathbf{X}_t = (X_t^{(A)}, X_t^{(B)})$  where  $X_t^{(A)} \in \{0, 1, 2, \dots\}$  and  $X_t^{(B)} \in \{0, 1, 2, \dots\}$  are the numbers of customers at service station  $A$  and  $B$  either in service or waiting in line.

- (i) Write down the transition rates  $\alpha((i, j), (k, l))$  for the Markov chain.
- (ii) Use (a) to show that if  $\lambda < \mu_A$  and  $\lambda < \mu_B$  the Markov chain is positive recurrent and compute the stationary distribution.

**Problem 3** Let  $Y$  be a random variable taking values in  $\{\dots, -2, -1, 0, 1\}$  and with p.d.f.  $P\{Y = j\} = p(j)$ . Consider the following Markov chain

$$X_{n+1} = \max\{X_n + Y, 0\}.$$

$X_n$  is a random walk with increments given by  $Y$  but the walk cannot jump below 0. The transition probabilities are

$$\begin{aligned} P(i, j) &= p(j - i), j > 0 \\ P(i, 0) &= \sum_{j \leq 0} p(j - i) \end{aligned}$$

- (a) Let  $q(n) = p(1 - n)$ . Show that if  $E[Y] < 0$  then there exists  $\alpha < 1$  such that

$$\alpha = \sum_{n=0}^{\infty} q(n)\alpha^n.$$

- (b) Use the result in part (a) to show that if  $E[Y] < 0$  then  $X_n$  is positive recurrent and compute the stationary distribution  $\pi$ . *Hint:* Study the equation  $\pi P = \pi$ .