1. Study the stability of the critical points of the equation
\[
\begin{align*}
  x_1' &= (x_1 - x_2)(1 - x_1 - x_2)/3, \\
  x_2' &= x_1(2 - x_2).
\end{align*}
\]

2. Consider the FitzHugh-Nagumo equation
\[
\begin{align*}
  x_1' &= f_1(x_1, x_2) = g(x_1) - x_2, \\
  x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2,
\end{align*}
\]
where \(\sigma\) and \(\gamma\) are positive constants and the function \(g\) is given by \(g(x) = -x(x - 1/2)(x - 1)\). Show that as the ratio \(\sigma/\gamma\) decreases the system undergoes a bifurcation from one equilibrium state to three equilibrium states. Compute the critical points and determine their stability properties. Some of the computations are lengthy and you might want to use a geometric argument to determine stability: just look at the directions of the vector field! It is also good idea to make a graph of the orbits before and after the bifurcation (use matlab or mathematica).

3. Let \(x, y \in \mathbb{R}^n\) and consider the Hamiltonian
\[
H(x, y) = \sum_{i=1}^n \frac{y_i^2}{2} + W(x)
\]
where \(W(x)\) is of class \(C^2\). Assume that \(a\) is a nondegenerate critical point of \(W\), i.e.
\[
\nabla W(a) = 0 \quad \text{and} \quad \det \left( \frac{\partial^2 W}{\partial q_i \partial q_j}(a) \right) \neq 0.
\]
and consider the Hamiltonian equation
\[
x'' = -\nabla W(x).
\]
Using linearization show that \((a, 0)\) is an unstable critical point if \(a\) is local maximum or a saddle point of \(W\). Hint: Study the eigenvalues!

4. Let \(f : \mathbb{R}^n \to \mathbb{R}^n\) be of class \(C^1\). Assume that the solutions of \(x' = f(x)\) exists for all \(t \in \mathbb{R}\) and denote by \(\phi^t\) the corresponding flow \(\phi^t(x) = x(t, 0, x)\).

(a) Prove Liouville Theorem
\[
\det \left( \frac{\partial \phi^t}{\partial x} \right) = \exp \left( \int_0^t \text{div} f(\phi^s(x)) \, ds \right).
\]
where \(\text{div} f(x) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x)\). Hint: Use Liouville theorem for linear ODE and the variational equation.

(b) Show that \(\phi^t\) is volume preserving if and only if \(\text{div} f = 0\). Hint: A map \(T : \mathbb{R}^n \to \mathbb{R}^n\) is volume preserving if \(\text{vol}(T(A)) = \text{vol}(A)\) for all sets \(A\) compact for which \(\partial A\) is negligible (for Riemann integral) or Lebesgue measurable (if you prefer Lebesgue integral).
5. Consider the system of equations

\[
\begin{align*}
x'_1 &= -x_1, \\
x'_2 &= -x_2 + x_1^2, \\
x'_3 &= x_3 + x_2^2.
\end{align*}
\]  

(6)

Compute the first four approximations \(u^{(j)}(t, a)\) for the functions defining the stable manifold. Show that \(u^{(3)}(t, a) = u^{(4)}(t, a)\) and thus \(u(t, a) = u^{(3)}(t, a)\). Determine then the stable and unstable manifolds \(W^s\) and \(W^u\).

6. Consider the equation (see the example in class)

\[
\begin{align*}
x' &= x^2, \\
y' &= -y.
\end{align*}
\]  

(7)

Show that for any \(c \in \mathbb{R}\), the function

\[
h_c(x) = \begin{cases} 
    ce^{1/x} & \text{for } x < 0 \\
    0 & \text{for } x \geq 0 
\end{cases}
\]  

(8)

determines a center manifold for this system. Graph \(h_c(x)\) for various \(c\).

7. Using center manifolds, determine the qualitative behavior near the origin for the equation

\[
\begin{align*}
x' &= xy, \\
y' &= -y - x^2.
\end{align*}
\]  

(9)

8. Consider the equation

\[
x' = A(t)x + g(t, x),
\]

(10)

where \(A(t)\) is continuous and periodic of period \(p\) and \(g\) is continuous, satisfy a local Lipschitz condition, and

\[
\lim_{\|x\| \to 0} \sup_{t > t_0} \frac{\|g(t, x)\|}{\|x\|} = 0
\]  

(11)

Let \(R\) be the matrix given in Floquet Theorem. Show that 0 is stable if all the negative eigenvalues of \(R\) have negative real part and is unstable if at least one eigenvalue of \(R\) has positive real part. \textit{Hint: } Consider the change of variables \(x = P(t)y\) where \(P(t)\) is the periodic matrix given in Floquet Theorem.

9. Determine the stability of the \((0, 0)\) solution of

\[
\begin{align*}
x'_1 &= x_1x_2^2 - 2x_2, \\
x'_2 &= x_1 - x_1^2x_2.
\end{align*}
\]
10. Determine the stability of the \((0,0)\) solution of
\[
x' = -x + y + xy, \\
y' = x - y - x^3 - y^3.
\]

11. Consider the equation
\[
x' = Ax + f(x), \tag{12}
\]
where \(f\) is locally Lipschitz and satisfy the condition
\[
\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0. \tag{13}
\]
Assume that \(A\) is diagonalizable and that all its eigenvalues \(\lambda_1 \leq \cdots \leq \lambda_n\) are real. Show directly by using Liapunov functions, that
(a) If all the eigenvalues of \(A\) are negative, then 0 is asymptotically stable.
(b) If, for \(0 < p < n\), \(\lambda_1 \leq \cdots \lambda_p < 0\) and \(0 < \lambda_{p+1}, \cdots < \lambda_n\) then 0 is unstable.

12. Consider the system
\[
x' = 2y(z - 1), \\
y' = -x(z - 1) \\
z' = xy. \tag{14}
\]
(a) Show that \((0,0,1)\) is stable.
(b) Is it asymptotically stable?