Differential Equations
and
Dynamical Systems

Classnotes for Math 645

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Chapter 1

Existence and Uniqueness

1.1 Introduction

An *ordinary differential equation (ODE)* is given by a relation of the form

$$F(t, x, x', x'', \cdots, x^{(m)}) = 0, \tag{1.1}$$

where $t \in \mathbf{R}$, $x, x', \cdots, x^{(m)} \in \mathbf{R}^n$ and the function F is defined on some open set of $\mathbf{R} \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$. A function $x : I \to \mathbf{R}^n$, where I is an interval in $\mathbf{R}$, is a solution of (1.1) if $x(t)$ is of class $\mathcal{C}^m$ (i.e., m-times continuously differentiable) and if

$$F(t, x(t), x'(t), x''(t), \cdots, x^{(m)}(t)) = 0 \quad \text{for all } t \in I. \tag{1.2}$$

We say that the ODE is of order m if the maximal order of the derivative occurring in (1.1) is m.

Example 1.1.1 Clairaut equation (1734) Let us consider the first order equation

$$x - tx' + f(x') = 0, \tag{1.3}$$

where f is some given function. It is given, in implicit form, by a nonlinear equation in x'. It is easy to verify that the lines $x(t) = Ct - f(C)$ are solutions of (1.3) for any C. Consider for example $f(z) = z^2 + z$, then one sees easily that given a point (t_0, x_0) there exists either 0 or 2 such solutions passing by the point (t_0, x_0) (see Figure 1.1).

As we see from this example, it is in general very difficult to obtain results on the uniqueness or existence of solutions for general equations of the form (1.1). We will therefore restrict ourselves to situations where (1.1) can be solved as a function of $x^{(m)}$,

$$x^{(m)} = g(t, x, x', \cdots, x^{(m-1)}). \tag{1.4}$$

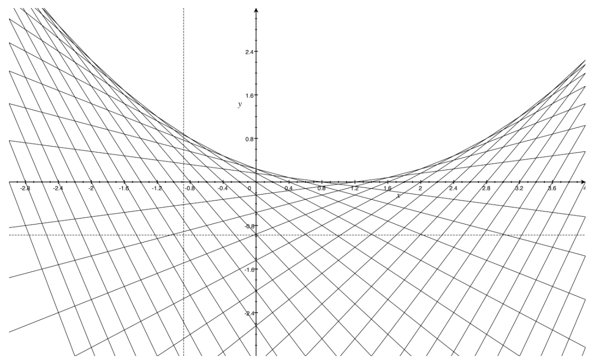

Figure 1.1: Some solutions for Clairault equation for $f(z) = z^2 + z$.

Such an equation is called an *explicit* ODE of order m. One can always reduce an ODE of order m to a first order ODE for a vector in a space of larger dimension. For example we can introduce the new variables

$$x_1 = x\,, x_2 = x'\,, x_3 = x'' \cdots, x_m = x^{(m-1)}\,, \tag{1.5}$$

and rewrite (1.4) as the system

$$\begin{aligned} x_1' &= x_2\,, \\ x_2' &= x_3\,, \\ &\vdots \\ x_{m-1}' &= x_m\,, \\ x_m' &= g(t, x_1, x_2, \cdots, x_m)\,. \end{aligned} \tag{1.6}$$

This is an equation of order 1 for the supervector $x = (x_1, \cdots, x_m) \in \mathbf{R}^{nm}$ (each x_i is in $\mathbf{R}^n$) and it has the form $x' = f(t, x)$. Therefore, in general, it is sufficient to consider the case of first order equations (m=1).

If f does not depend explicitly on t, i.e., $f(t, x) = f(x)$, the ODE $x' = f(x)$ is called *autonomous*. The function $f : U \to \mathbf{R}^n$, where U is an open set of $\mathbf{R}^n$, defines a *vector field*. A solution of $x' = f(x)$ is then a parametrized curve $x(t)$ which is tangent to the vector field $f(x)$ at any point, see figures 1.2 and (1.3).

Note a non-autonomous ODE $x' = f(t, x)$ with $x \in \mathbf{R}^n$ can be written as an autonomous ODE in $\mathbf{R}^{n+1}$ by setting

$$y = \begin{pmatrix} x \\ t \end{pmatrix} \qquad y' = \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} f(t, x) \\ 1 \end{pmatrix} \equiv F(y)\,. \tag{1.7}$$

Example 1.1.2 Predator-Prey equation Let us consider the equation

$$x' = x(\alpha - \beta y), \quad y' = y(\gamma x - \delta), \tag{1.8}$$

where $\alpha, \beta, \gamma, \delta$ are given positive constants. Here $x(t)$ is the population of the preys and $y(t)$ is the population of the predators. If the population of predators y is below the threshold α/β then x is increasing while if y is above α/β then x is decreasing. The opposite holds for the population y. In order to study the solutions, let us divide the first equation, by the second one and consider x as a function of y. We obtain

$$\frac{dx}{dy} = \frac{x}{y}\frac{(\alpha - \beta y)}{(\gamma x - \delta)} \quad \text{or} \quad \frac{(\gamma x - \delta)}{x}dx = \frac{(\alpha - \beta y)}{y}dy. \tag{1.9}$$

Integrating gives

$$\gamma x - \delta \log x = \alpha \log y - \beta y + \text{Const.} \tag{1.10}$$

One can verify that the level curves (1.10) are closed bounded curves and each solution $(x(t), y(t))$ stays on a level curve of (1.10) for any $t \in \mathbf{R}$. This suggests that the solutions are periodic (see Figure 1.2).

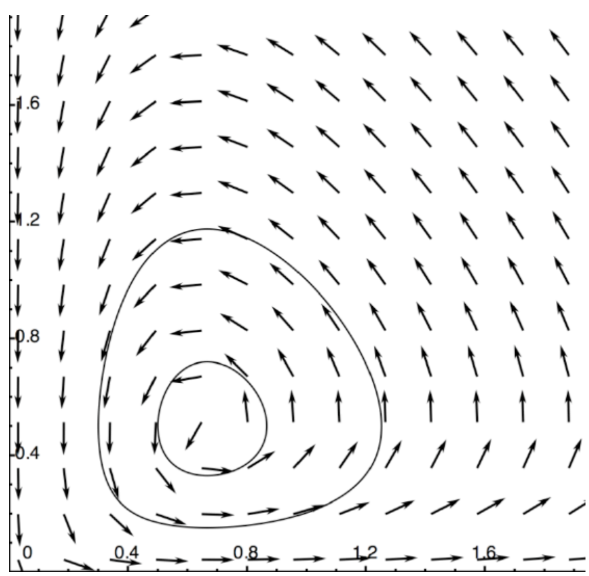

Figure 1.2: The vector field for the predator-prey equation with $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 2$ and the solutions passing through the point $(1, 1)$ and $(0.5, 0.5)$.

Example 1.1.3 van der Pol equation. The van der Pol equation

$$x'' = \epsilon(1 - x^2)x' - x. \tag{1.11}$$

It can written as a first order system by setting $y = x'$

$$\begin{aligned} x' &= y\,, \\ y' &= \epsilon(1-x^2)y - x\,. \end{aligned} \tag{1.12}$$

It is a perturbation of the harmonic oscillator ($\epsilon = 0$) $x'' + x = 0$ whose solutions are the periodic solution $x(t) = A\cos(t-\phi)$ and $y(t) = x'(t) = -A\sin(t-\phi)$ (circles). When $\epsilon > 0$ one observes that one periodic solution survives which is the deformation of a circle of radius 2 and all other solution are attracted to this periodic solution (limit cycle), see Figure 1.3.

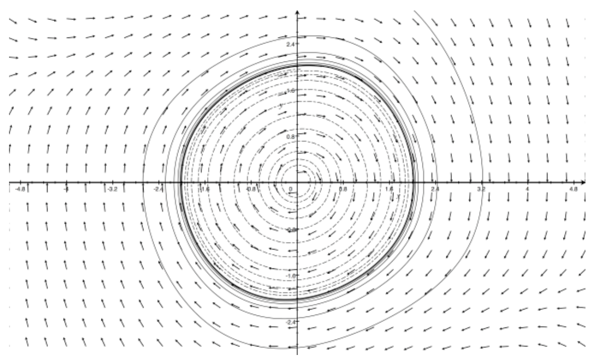

Figure 1.3: The vector field for the van der Pol equation with $\epsilon = 0.1$ as well as two solutions passing through the points $(.1, .2)$ and $(2, 3)$.

We will discuss these examples in more details later. For now we observe that, in both cases, the solutions curves never intersect. This means that there are never two solutions passing by the same point. Our first goal will be to find sufficient conditions for the problem

$$x' = f(t,x)\,, \quad x(t_0) = x_0\,, \tag{1.13}$$

to have a unique solution. We say that t_0 and x_0 are the *initial values* and the problem (1.13) is called a Cauchy Problem or an initial value problem (IVP).

1.2 Banach fixed point theorem

We will need some (simple) tools of functional analysis. Let E be a vector space with addition $+$ and multiplication by scalar λ in $\mathbf{R}$ or $\mathbf{C}$. A *norm* on E is a map $\|\cdot\| : E \to \mathbf{R}$ which satisfies the following three properties

- **N1** $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$,

- **N2** $\|\lambda x\| \, = \, |\lambda| \|x\|$,
- **N3** $\|x+y\| \, \leq \, \|x\| + \|y\|$ (triangle inequality) .

A vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*.

In a normed vector space E we can define the convergence of sequence $\{x_n\}$. We say that the sequence $\{x_n\}$ *converges* to $x \in E$, if for any $\epsilon > 0$, there exists $N \geq 1$ such that, for all $n \geq N$, we have $\|x_n - x\| \leq \epsilon$.

We say that $\{x_n\}$ is a *Cauchy sequence* if for any $\epsilon > 0$, there exists $N \geq 1$ such that, for all $n, m \geq N$, we have $\|x_n - x_m\| \leq \epsilon$.

Definition 1.2.1 A normed vector space E is said to be *complete* if every Cauchy sequence in E converges to an element of E. A complete normed vector space E is called a *Banach space.*

Let $\|\cdot\|$ and $\|\cdot\|_\star$ denote two norms on the vector space E. We say that the norms $\|\cdot\|$ and $\|\cdot\|_\star$ are *equivalent* if there exist positive constants c and C such that

$$c\|x\| \leq \|x\|_\star \leq C\|x\| \quad \text{for all } x \in E \, .$$

It is easy to check that the equivalence of norm defines an equivalence relation. Furthermore if a Cauchy sequence for a norm $\|\cdot\|$ is also a Cauchy sequence for an equivalent norm $\|\cdot\|_\star$.

Example 1.2.2 The vector space $E = \mathbf{R}^n$ or $\mathbf{C}^n$ with the euclidean norm $\|x\|_2 = (\sum_i x_i^2)^{1/2}$ is a Banach space. Other examples of norms are $\|x\|_1 = \sum_i |x_i|$ or $x_\infty = \sup_i |x_i|$. In any case $\mathbf{R}^n$ or $\mathbf{C}^n$ equipped with any norm is a Banach space, since all norm are equivalent in a finite-dimensional space (see exercises).

The previous example shows that for finite dimensional vector spaces the choice of a norm does not matter much. For infinite-dimensional vector spaces the situation is very different as the following example demonstrate.

Proposition 1.2.3 *Let*

$$\mathcal{C}([0,1]) \, = \, \{f \, : \, [0,1] \to \mathbf{R}^n \, ; \, f \text{ continuous}\} \, . \tag{1.14}$$

With the norm

$$\|f\|_\infty \, = \, \sup_{t\in[0,1]} |f(t)| \, . \tag{1.15}$$

$\mathcal{C}([0,1])$ *is a Banach space. With either of the norms*

$$\|f\|_1 \, = \, \int_0^1 |f(t)| \, dt \, , \quad \text{or} \quad \|f\|_2 \, = \, \left(\int_0^1 |f(t)|^2 \, dt\right)^{1/2} \, , \tag{1.16}$$

$\mathcal{C}([0,1])$ *is not complete.*

Proof: We let the reader verify that $\|f\|_1$, $\|f\|_2$, and $\|f\|_\infty$ are norms.

Let $\{f_n\}$ be a Cauchy sequence for the norm $\|\cdot\|_\infty$. We have then

$$|f_n(t) - f_m(t)| \leq \|f_n - f_m\|_\infty \leq \epsilon \quad \text{for all } n, m \geq N\,. \tag{1.17}$$

This implies that, for any t, $\{f_n(t)\}$ is a Cauchy sequence in $\mathbf{R}$ which is complete. Therefore $\{f_n(t)\}$ converges to an element of $\mathbf{R}$ which we call $f(t)$. It remains to show that the function $f(t)$ is continuous. Taking the limit $m \to \infty$ in (1.17), we have

$$|f_n(t) - f(t)| \leq \epsilon \quad \text{for all } n \geq N\,, \tag{1.18}$$

where N depends on ϵ but not on t. This means that $f_n(t)$ converges uniformly to $f(t)$ and therefore $f(t)$ is continuous.

Let us consider the sequence $\{f_n\}$ of piecewise linear continuous functions, where $f_n(t) = 0$ on $[0, 1/2 - 1/n]$ and $f_n(t) = 1$ on $[1/2 + 1/n, 1]$ and linearly interpolating in between. One verifies easily that for any $m \geq n$ we have $\|f_n - f_m\|_1 \leq 1/n$ and $\|f_n - f_m\|_2 \leq 1/\sqrt{n}$. Therefore $\{f_n\}$ is a Cauchy sequence. But the limit function is not continuous and therefore the sequence does not converge in $\mathcal{C}([0, 1])$. ∎

We have also

Proposition 1.2.4 *Let X be an arbitrary set and let us consider the space*

$$\mathcal{B}(X) = \{f : X \to \mathbf{R}\,;\, f \text{ bounded}\}\,. \tag{1.19}$$

with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|\,. \tag{1.20}$$

Then $\mathcal{B}(X)$ is a Banach space.

Proof: The proof is almost identical to the first part the previous proposition and is left to the reader.

In a Banach space E we can define basic topological concepts as in $\mathbf{R}^n$.

- A *open ball* of radius r and center a is the set $B_\epsilon(a) = \{x \in E\,;\, \|x - a\| < r\}$.
- A *neighborhood of* a is a set V such that $B_\epsilon(a) \subset V$ for some $\epsilon > 0$.
- A set $U \subset E$ is *open* if U is a neighborhood of each of its element, i.e., for any $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.
- A set $V \subset E$ is *closed* if the limit of any convergent sequence $\{x_n\}$ is in V.
- A set K is compact if any sequence $\{x_n\}$ with $x_n \in K$ has a subsequence which converges in K.

- Let E and F be two Banach spaces and $U \subset E$. A function $f : U \to F$ is continuous at $x_0 \in U$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $x \in U$ and $\|x - x_0\| < \delta$ implies that $\|f(x) - f(x_0)\| < \epsilon$.

- The map $x \mapsto \|x\|$ is a continuous function of E to $\mathbf{R}$, since $|\|x\| - \|x_0\|| \leq \|x - x_0\|$ by the triangle inequality.

Certain properties which are true in finite dimensional Banach spaces are not true in infinite dimensional Banach spaces such as the function spaces we have considered in Propositions 1.2.3 and 1.2.4. For example we show that

- The closed ball $\{x \in E\,;\, \|x\| \leq 1\}$ is not necessarily compact.

- Two norms on a Banach space are not always equivalent.

- The theorem of Bolzano-Weierstrass which says each bounded sequence has a convergent subsequence is not necessarily true.

- The equivalence of K compact and K closed and bounded is not necessarily true.

The proposition 1.2.3 shows that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent. For, if they were equivalent, any Cauchy sequence for $\|\cdot\|_1$ would be a Cauchy sequence $\|\cdot\|_\infty$. But we have constructed explicitly a Cauchy sequence for $\|\cdot\|_1$ which is not a Cauchy sequence for $\|\cdot\|_\infty$. Let us consider the Banach space $\mathcal{B}([0,1])$ and let $f_n(t)$ to be equal to 1 if $1/(n+1) < t \leq 1/n$ and 0 otherwise. We have $\|f_n\|_\infty = 1$ for all n and $\|f_n - f_m\|_\infty = 1$ for any n, m. Therefore $\{f_n\}$ cannot have a convergent subsequence. This shows at the same time, that the unit ball is not compact, that Bolzano-Weierstrass fails, and that closed bounded sets are not necessarily compact.

Let us suppose that we want to solve a nonlinear equation in a Banach space E. Let f be a function from E to E then we might want to solve

$$f(x) \;=\; x \quad \text{find a fixed point of } f\,. \tag{1.21}$$

The next theorem will provide a sufficient condition for the existence of a fixed point.

Theorem 1.2.5 (Banach Fixed Point Theorem (1922)) *Let E be a Banach space, $D \subset E$ closed and $f : D \to E$ a map which satisfies*

1. $f(D) \subset D$.

2. *f is a contraction on D, i.e., there exists $\alpha < 1$ such that,*

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|\,, \quad \text{for all } x, y \in D\,. \tag{1.22}$$

Then f has a unique fixed point x in D, $f(x) = x$.

Proof: We first show uniqueness. Let us suppose that there are two fixed points x and y, i.e., $f(x) = x$ and $f(y) = y$. Since f is a contraction we have

$$\|x - y\| = \|f(x) - f(y)\| \leq \alpha \|x - y\| \tag{1.23}$$

with $\alpha < 1$, this is possible only if $x = y$.

To prove the existence we choose an arbitrary $x_0 \in D$ and we consider the iteration $x_1 = f(x_0), \cdots, x_{n+1} = f(x_n), \cdots$. Since $f(D) \subset D$ this implies that $x_n \in D$ for any n. Let us show that $\{x_n\}$ is a Cauchy sequence. We have $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq \alpha \|x_n - x_{n-1}\|$. Iterating this inequality we obtain

$$\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\| \,. \tag{1.24}$$

If $m > n$ this implies that

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots \|x_{n+1} - x_n\| \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots \alpha^n\right) \|x_1 - x_0\| \\ &\leq \frac{\alpha^n}{1-\alpha} \|x_1 - x_0\| \,. \end{aligned} \tag{1.25}$$

Therefore $\{x_n\}$ is a Cauchy sequence since $\alpha^n \to 0$. Since E is a Banach space, this sequence converges to $x \in E$. The limit x is in D since D is closed. Since f is a contraction, it is continuous and we have

$$x = \lim_{n\to\infty} x_{n+1} = f(\lim_{n\to\infty} x_n) = f(x) \,, \tag{1.26}$$

i.e., x is a fixed point of f. ∎

The proof of the theorem is constructive and provides the following algorithm to construct a fixed point.

Method of successive approximations: To solve $f(x) = x$

- Choose an arbitrary x_0.
- Iterate: $x_{n+1} = f(x_n)$.

Even if the hypotheses of the theorem are difficult to check, one might apply this algorithm. If the algorithm converges this gives a fixed point, although not necessarily a unique one.

Example 1.2.6 The function $f(x) = \cos(x)$ has a fixed point on $D = [0, 1]$. By the mean value theorem there is $\xi \in (x, y)$ such that $\cos(x) - \cos(y) = \sin(\xi)(y - x)$, thus $|\cos(x) - \cos(y)| \leq \sup_{t\in[0,1]} |\sin(t)||x - y| \leq \sin(1)|x - y|$, and $\sin(1) < 1$. One observes a quite rapid convergence to the solution $0.7390\cdots$. For example we have $x_0 = 0$, $x_1 = 1$, $x_2 = 0.5403$, $x_2 = 0.8575$, $x_3 = 0.6542$, $x_4 = 0.7934$, $x_5 = 0.7013$, $x_6 = 0.7639$, $\cdots$.

Example 1.2.7 Consider the Banach space $\mathcal{C}([0,1])$ with the norm $\|\cdot\|_\infty$. Let $f \in \mathcal{C}([0,1])$ and let $k(t,s)$ be a function of 2 variables continuous on $[0,1]\times[0,1]$. Consider the fixed point problem

$$x(t) \;=\; f(t) + \lambda \int_0^1 k(t,s)x(s)\,ds\,. \tag{1.27}$$

We assume that λ is such that $\alpha \equiv |\lambda| \sup_{0\le t\le 1} \int_0^1 |k(t,s)|\,ds < 1$. Consider the map $(Tx)(t) = f(t) + \lambda \int_0^1 k(t,s)y(s)$. The map $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ is well defined and one has the bound

$$|(Tx)(t) - Ty(t)| \;\le\; |\lambda| \int_0^1 |k(t,s)||x(s)-y(s)|\,ds \;\le\; \|x-y\|_\infty |\lambda| \sup_{0\le t\le 1} \int_0^1 |k(t,s)|\,ds\,. \tag{1.28}$$

Taking the supremum over t on the left side gives

$$\|Tx - Ty\|_\infty \;\le\; \alpha \|x-y\|_\infty\,, \tag{1.29}$$

so that T is a contraction. Hence the Banach fixed point theorem with $D = \mathcal{C}([0,1])$ implies the existence of a unique solution for (1.27). The method of successive approximation applies and the iteration is, $y_0(t) = f$ and

$$y_{n+1}(t) \;=\; f(t) + \lambda \int_0^1 k(t,s)y_n(s)\,ds\,. \tag{1.30}$$

1.3 Existence and uniqueness for the Cauchy problem

Let us consider the Cauchy problem

$$x'(t) = f(t, x(t))\,, \quad x(t_0) = x_0\,, \tag{1.31}$$

where $f : U \to \mathbf{R}^n$ (U is an open set of $\mathbf{R}\times\mathbf{R}^n$) is a continuous function. In order to find a solution we will rewrite (1.31) as a fixed point equation. We integrate the differential equation between t_0 an t, we obtain the *integral equation*

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\,ds\,. \tag{1.32}$$

Every solution of (1.31) is thus a solution of (1.32). The converse also holds. If $x(t)$ is a continuous function which verifies (1.32) on some interval I, then it is automatically of class $\mathcal{C}^1$ and it satisfies (1.31).

Let I be an interval and let us define the map $T : \mathcal{C}(I) \to \mathcal{C}(I)$ given by

$$(Tx)(t) \;=\; x_0 + \int_{t_0}^t f(s, x(s))\,ds\,. \tag{1.33}$$

The integral equation (1.32) can then be written as the fixed point equation

$$(Tx)(t) \;=\; x(t)\,, \tag{1.34}$$

i.e., we have transformed the differential equation (1.31) into a fixed point problem. The method of successive approximation for this problem is called

Picard-Lindelöf iteration:

$$\begin{aligned} x_0(t) &= x_0 \quad \text{(or any other function)}\,, \\ x_{n+1}(t) &= x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds\,. \end{aligned} \tag{1.35}$$

Example 1.3.1 Let us consider the Cauchy problem

$$x' = -x^2\,, \quad x(0) = 1\,. \tag{1.36}$$

The solution is $x(t) = \frac{1}{1+t}$. The Picard-Lindelöf iteration gives $x_0 = 1$, $x_1 = 1 - t$, $x_2 = 1 - t + t^2 - t^3/3$, and so on. One sees from Figure 1.4 that it converges in a suitable interval around 0 but diverges for larger values of t.

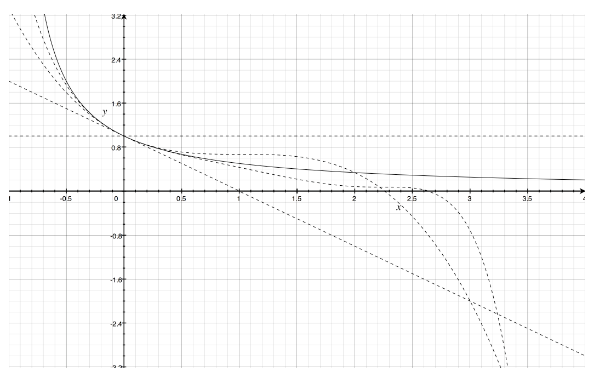

Figure 1.4: The first four iterations for the Picard Lindelöf iteration scheme for the Cauchy problem $x' = -x^2$, $x(0) = 1$.

The next result shows how to choose the interval I such that T maps a suitably chosen set D into itself. We have

Lemma 1.3.2 *Let $A = \{(t,x)\,;|t-t_0| \leq a\,,\, \|x-x_0\| \leq b\}$, $f : A \to \mathbf{R}^n$ be a continuous function with $M = \sup_{(t,x)\in A} |f(t,x)|$. We set $\alpha = \min(a, b/M)$. The map T given by (1.33) is well-defined on the set*

$$B = \{x : [t_0 - \alpha, t_0 + \alpha] \to \mathbf{R}^n\,,\, x \text{ continuous and } \|x(t) - x_0\| \leq b\}\,. \tag{1.37}$$

and it satisfies $T(B) \subset B$.

Proof: The lemma follows from the estimate

$$\|(Tx)(t) - x_0\| = \left\| \int_{t_0}^{t} f(s, x(s))\, ds \right\| \leq M|t - t_0| \leq M\alpha \leq b\,. \tag{1.38}$$

∎

We say that a function $f : A \to \mathbf{R}^n$ (with A is in the previous lemma) satisfies a *Lipschitz condition* if

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \text{for all } (t,x),(t,y) \in A\,. \tag{1.39}$$

The constant L is called the *Lipschitz constant*.

Remark 1.3.3 In order to illustrate the meaning of condition (1.39), let us suppose that $f(t,x) = f(x)$ does not depend on t and that we have $\|f(x) - f(y)\| \leq L\|x - y\|$ whenever x and y are in the closed ball $\overline{B}_b(x_0)$. This clearly implies that f is continuous in $\overline{B}_b(x_0)$ and f is called *Lipschitz continuous*. The opposite does not hold, for example the function $f(x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous at 0.

If f is of class $\mathcal{C}^1$, then f is Lipschitz continuous. To see this consider the line $z(s) = x + s(y - x)$ which interpolates between x and y. We have

$$\begin{aligned} \|f(y) - f(x)\| &= \left\| \int_0^1 \frac{d}{ds} f(z(s))\, ds \right\| = \left\| \int_0^1 f'(z(s))(y - x)\, ds \right\| \\ &\leq \sup_{z \in \overline{B}_b(x_0)} \|f'(z)\| \|y - x\|\,, \end{aligned} \tag{1.40}$$

and therefore f is Lipschitz continuous with $L = \sup_{\{x\,,\,\|x - x_0\| \leq b\}} \|f'(x)\|$. On the other hand Lipschitz continuity does not imply differentiability as the function $f(x) = |x|$ demonstrates.

The condition (1.39) requires that $f(t,x)$ is Lipschitz continuous in x uniformly in t with $|t - t_0| \leq a$.

If $f(t,x)$ satisfy a Lipschitz condition we have, for any $t \in I = [t_0 - \alpha, t_0 + \alpha]$,

$$\begin{aligned} \|(Tx)(t) - (Tz)(t)\| &\leq \int_{t_0}^{t} \|f(t, x(t)) - f(t, z(t))\|\, dt \\ &\leq \int_{t_0}^{t} L\|x(t) - z(t)\|\, dt \\ &\leq \alpha L \sup_{t \in I} \|x(t) - z(t)\| \;\leq\; \alpha L\|x - z\|_\infty\,. \end{aligned} \tag{1.41}$$

Taking the supremum over t on the left side shows that $\|Tx - Tz\|_\infty \leq \alpha L\|x - z\|$. If $\alpha L < 1$ we can apply the Banach fixed point theorem to prove existence of a fixed point and show the existence and uniqueness for the solution of the Cauchy problem for t in some sufficiently small interval around t_0.

In fact one can omit the condition $\alpha L < 1$ by applying the method of successive approximation directly without invoking the Banach Fixed Point Theorem. This is the content of the following theorem which is the basic result on existence of *local solutions* for the Cauchy problem (1.31). Here local means that we show the existence only of $x(t)$ is in some interval around t_0.

Theorem 1.3.4 (Existence and uniqueness for the Cauchy problem) *Let* $A = \{(t,x)\,;\,|t-t_0| \le a\,,\ \|x-x_0\| \le b\}$ *and let us suppose that* $f\,:\,A \to \mathbf{R}^n$

- *is continuous,*
- *satisfies a Lipschitz condition.*

Then the Cauchy problem $x' = f(t,x)$*,* $x(t_0) = x_0$ *has a unique solution on* $I = [t_0-\alpha, t_0+\alpha]$*, where* $\alpha = \min(a, b/M)$ *with* $M = \sup_{(t,x)\in A} \|f(t,x)\|$.

Proof: We prove directly that the Picard-Lindelöf iteration converge uniformly on I to a solution of the Cauchy problem. In a first step we show, by induction, that

$$\|x_{k+1}(t) - x_k(t)\| \,\le\, ML^k \frac{|t-t_0|^{k+1}}{(k+1)!} \quad \text{for } |t-t_0| \le \alpha\,. \tag{1.42}$$

For $k=0$, we have $\|x_1(t) - x_0\| \,=\, \|\int_{t_0}^t f(s,x(s))\,ds\| \le M|t-t_0|$. Let us assume that (1.42) holds for $k-1$. Then we have

$$\begin{aligned}\|x_{k+1}(t) - x_k(t)\| \;&\le\; \int_{t_0}^t \|f(s,x_k(s)) - f(s,x_{k-1}(s)\|\,ds \,\le\, L\int_{t_0}^t \|x_k(s) - x_{k-1}(s)\|\,ds \\ &\le\; ML^k \int_{t_0}^t \frac{|s-t_0|^k}{k!}\,ds \,=\, ML^k \frac{|t-t_0|^{k+1}}{(k+1)!}\,. \end{aligned} \tag{1.43}$$

Using (1.42), we show that $\{x_k(t)\}$ is a Cauchy sequence for the norm $\|x\|_\infty = \sup_{t\in I}\|x(t)\|$. We have

$$\begin{aligned}\|x_{k+m}(t) - x_k(t)\| \;&\le\; \|x_{k+m}(t) - x_{k+m-1}(t)\| + \cdots + \|x_{k+1}(t) - x_k(t)\| \\ &\le\; \frac{M}{L}\left(\frac{L^{k+m}|t-t_0|^{k+m}}{(k+m)!} + \cdots + \frac{L^{k+1}|t-t_0|^{k+1}}{(k+1)!}\right) \\ &\le\; \frac{M}{L}\sum_{j=k+1}^{\infty} \frac{(L\alpha)^j}{j!}\,, \end{aligned} \tag{1.44}$$

and the right hand side is the reminder term of a convergent series and thus goes to 0 as k goes to ∞. The right hand side is independent of t so $\{x_k\}$ is a Cauchy sequence which converges uniformly to a continuous function $x : I \to \mathbf{R}^n$.

To show that $x(t)$ is a solution of the Cauchy problem we take the limit $n \to \infty$ in (1.35). The left side converges uniformly to $x(t)$. Since f is continuous and A is compact $f(t, x_k(t))$ converges uniformly to $f(t, x(t))$ on A. Thus one can exchange integral and the limit and $x(t)$ is a solution of the integral equation (1.32).

It remains to prove uniqueness of the solution. Let $x(t)$ and $z(t)$ be two solutions of (1.32). By recurrence we show that

$$\|x(t) - y(t)\| \leq 2ML^k \frac{|t - t_0|^{k+1}}{(k+1)!} . \tag{1.45}$$

We have $x(t) - y(t) = \int_{t_0}^{t} (f(s, x(s)) - f(s, y(s)))\, ds$ and therefore $\|x(t) - y(t)\| \leq 2M|t - t_0|$ which (1.45) for $k = 0$. If (1.45) holds for $k - 1$ we have

$$\begin{aligned} \|x(t) - y(t)\| \leq \int_{t_0}^{t} L\|x(s) - y(s)\|\, ds &\leq 2ML^k \int_{t_0}^{t} \frac{|s - t_0|^k}{k!}\, dt \\ &\leq 2ML^k \frac{|t - t_0|^{k+1}}{(k+1)!} , \end{aligned} \tag{1.46}$$

and this proves (1.45). Since this holds for all k, this shows that $x(t) = y(t)$. ∎

1.4 Peano Theorem

In the previous section we established a local existence result by assuming a Lipschitz condition. Simple examples show that this condition is also necessary.

Example 1.4.1 Consider the ODE

$$x' = 2\sqrt{|x|} . \tag{1.47}$$

We find that $x(t) = (t - c)^2$ for $t > c$ and $x(t) = -(c - t)^2$ for $t < c$ is a solution for any constant c. But $x(t) \equiv 0$ is also a solution. The Cauchy problem with, say, $x(0) = 0$ has infinitely many solutions. For $t > 0$, $x(t) \equiv 0$ is one solution, $x = t^2$ is another solution, and more generally $x(t) = 0$ for $0 \leq t \leq c$ and then $x(t) = (t - c)^2$ for $t \geq c$ is also a solution for any c. This phenomenon occur because $\sqrt{|x|}$ is not Lipschitz continuous at $x = 0$.

We are going to show that, without Lipschitz condition, we can still obtain existence of solutions, but not uniqueness. Instead of using the Picard-Lindelöf iteration we are using another approximation scheme. It turns out to be the simplest algorithm used for numerical approximations of ODE's.

Euler polygon (1736) Fix some $h \neq 0$, the idea is to approximate the solution locally by $x(t+h) \simeq x(t)+hf(t,x(t))$. Let us consider now the sequence $\{t_n, x_n\}$ given recursively by

$$t_{n+1} = t_n + h\,, \quad x_{n+1} = x_n + hf(t_n, x_n)\,. \tag{1.48}$$

We then denote by $x_h(t)$ the piecewise linear function which passes through the points (t_n, x_n). It is called the Euler polygon and is an approximation to the solution of the Cauchy problem.

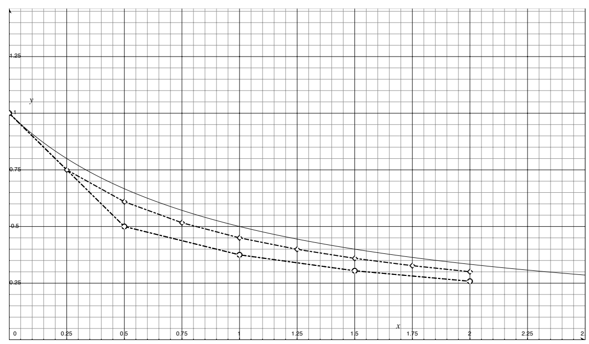

Figure 1.5: Euler polygons for $x' = -x^2$ for h=0.5 and $h = 0.25$.

Lemma 1.4.2 *Let $A = \{(t,x)\,;|t-t_0| \le a\,,\,\|x-x_0\| \le b\}$, and $f : A \to \mathbf{R}^n$ be a continuous function with $M = \sup_{(t,x)\in A}\|f(t,x)\|$. We set $\alpha = \min(a, b/M)$. If $h = \pm\alpha/N$, N an integer, the Euler Polygon satisfies $(t, x_h(t)) \in A$ for $t \in [t_0-\alpha, t_0+\alpha]$ and we have the bound*

$$\|x_h(t) - x_h(t')\| \le M|t-t'| \quad \text{for } t, t' \in [t_0-\alpha, t_0+\alpha]\,. \tag{1.49}$$

Proof: Let us consider first the interval $[t_0, t_0+\alpha]$ and choose $h > 0$. We show first, by induction that $(t_n, x_n) \in A$ for $n = 0, 1, \cdots, N$. We have $\|x_n - x_{n-1}\| \le hM$ and so $\|x_n - x_0\| \le nhM \le \alpha M \le b$ if $n \le N$. Since $x_h(t)$ is piecewise linear $(t, x_h(t)) \in A$ for any $t \in [t_0, t_0+\alpha]$. The estimate (1.49) follows from the fact that the slope of $x_h(t)$ is nowhere bigger than M. On $[t_0-\alpha, t_0]$ the argument is similar. ∎

Definition 1.4.3 A family of functions $f_j : [a,b] \to \mathbf{R}^n$, $j = 1, 2, \cdots$, is *equicontinuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for, for all j, $|t-t'| < \delta$ implies that $\|f_j(t) - f_j(t')\| \le \epsilon$.

Equicontinuity means that all the functions f_j are uniformly continuous and that, moreover, δ can be chosen to depend only on ϵ, but *not* on j. The estimate (1.49) shows that the family $x_h(t)$, with $h = \alpha/N$, $N = 1, 2, \cdots$, is equicontinuous.

Theorem 1.4.4 (Arzelà-Ascoli 1895) *Let $f_j : [a,b] \to \mathbf{R}^n$ be a family of functions such that*

- *$\{f_j\}$ is equicontinuous.*
- *For any $t \in [a,b]$, there exists $M(t) \in \mathbf{R}$ such that $\sup_j \|f_j(t)\| \le M(t)$.*

Then the family $\{f_j\}$ has a convergent subsequence $\{g_n\}$ which converges uniformly to a continuous function g on $[a,b]$.

Remark 1.4.5 As we have seen a bounded closed set in $\mathcal{C}([a,b])$ is not always compact. The Arzelà-Ascoli theorem shows that a bounded set of equicontinuous function is a compact set in $\mathcal{C}([a,b])$ and thus it can be seen a generalization of Bolzano-Weierstrass to $\mathcal{C}([a,b])$.

Proof: The subsequence is constructed proof via a trick which is referred to as "diagonal subsequence". The set of rational numbers in $[a,b]$ is countable and we write it as $\{t_1, t_2, t_3, \cdots\}$. Consider the sequence $\{f_j(t_1)\}$, by assumption it is bounded in $\mathbf{R}^n$ and, by Bolzano-Weierstrass, it has a convergent subsequence which we denote by $\{f_{1i}(t_1)\}_{i\ge 1}$ and therefore

$$f_{11}(t), f_{12}(t), f_{13}(t)\cdots \quad \text{converges for } t = t_1 . \tag{1.50}$$

Consider next the sequence $\{f_{1i}(t_2)\}_{i\ge1}$. Again, by Bolzano-Weierstrass, this sequence has a convergent subsequence denoted by $\{f_{2i}(t_2)\}_{i\ge1}$. We have

$$f_{21}(t), f_{22}(t), f_{23}(t)\cdots \quad \text{converges for } t = t_1, t_2 . \tag{1.51}$$

After n steps we find a sequence $\{f_{ni}(t)\}_{i\ge1}$ of $\{f_j(t)\}$ such that

$$f_{n1}(t), f_{n2}(t), f_{n3}(t)\cdots \quad \text{converges for } t = t_1, t_2, \cdots, t_n . \tag{1.52}$$

Next we consider the diagonal sequence $g_n(t) = f_{nn}(t)$. This sequence converges for any t_l, since $\{g_n(t_l) = f_{nn}(t_l)\}_{n\ge l}$ is a subsequence of $\{f_{ln}(t_l)\}_{n\ge l}$ which converges.

By equicontinuity, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $n \ge 1$, $|t-t'| < \delta$ implies that $\|g_n(t) - g_n(t')\| < \epsilon$. Let us choose rational points $t_1, t_2, \cdots t_{q-1}$ such that $a = t_0 < t_1 < \cdots < t_{q-1} < t_q = b$ and $t_{i+1} - t_i < \delta$. For $t \in [t_l, t_{l+1}]$ we have

$$\|g_n(t) - g_m(t)\| \le \|g_n(t) - g_n(t_l)\| + \|g_n(t_l) - g_m(t_l)\| + \|g_m(t) - g_m(t_l)\| . \tag{1.53}$$

By equicontinuity $\|g_n(t) - g_n(t_l)\|$ and $\|g_m(t) - g_m(t_l)\|$ are smaller than ϵ. By the convergence of $\{g_n(t_l)\}$ there exists $N(l)$ such that $\|g_n(t_l) - g_m(t_l)\| \le \epsilon$ if $n, m \ge N_l$. If we choose $N = \max_l N_l$ we have that $\|g_n(t) - g_m(t)\| \le 3\epsilon$ for all $t \in [a,b]$ and $n, m \ge N$. This shows that $g_n(t)$ converges uniformly to some $g(t)$ which is then continuous. ∎

From this we obtain

Theorem 1.4.6 (Peano 1890) *Let* $A = \{(t,x)\,;|t-t_0| \le a\,,\, \|x-x_0\| \le b\}$*,* $f : A \to \mathbf{R}^n$ *a continuous function with* $M = \sup_{(t,x)\in A} \|f(t,x)\|$*. Set* $\alpha = \min(a, b/M)$*. The Cauchy problem (1.31) has a solution on* $[t_0-\alpha, t_0+\alpha]$*.*

Proof: Let us consider the Euler polygons with $h = \alpha/N$, $N = 1, 2, \cdots$. The sequence is bounded since $\|x_h(t) - x_0\| \le M|t-t_0| \le M\alpha$ and equicontinuous by Lemma 1.4.2. By Arzelà-Ascoli Theorem, the family $x_h(t)$ has a subsequence which converges uniformly to a continuous function $x(t)$ on $[t_0-\alpha, t_0+\alpha]$. It remains to show that $x(t)$ is a solution.

Let $t \in [t_0, t_0+\alpha]$ and let (t_n, x_n) the approximation obtained by Euler method for $x_h(t)$. If $t \in [t_l, t_{l+1}]$ we have

$$x_h(t) - x_0 \;=\; hf(t_0,x_0) + hf(t_1,x_1) + \cdots + hf(t_{l-1},x_{l-1}) + (t-t_l)f(t_l,x_l)\,. \quad (1.54)$$

Since $f(t, x(t))$ is a continuous function of t it is Riemann integrable and, using a Riemann sum with left-end points have

$$\begin{aligned}\int_{t_0}^{t} f(s,x(s))\,ds \;&=\; hf(t_0,x(t_0)) + hf(t_1,x(t_1)) + \cdots \\ &\quad \cdots + hf(t_{l-1},x(t_{l-1})) + (t-t_l)f(t_l,x(t_l)) + r(h)\,, \quad (1.55)\end{aligned}$$

where $\lim_{h\to 0}\|r(h)\| = 0$. By the uniform continuity of f on A and the uniform convergence of the subsequence of $\{x_h(t)\}$ to $x(t)$ we have that $\|f(t,x_h(t) - f(t,x(t))\| \le \epsilon$ if h is sufficiently small (and h is such that x_h belongs to the convergent subsequence). Using that $x_h(t_j) = x_j$ and subtracting (1.55) from (1.54) we find that

$$\|x_h(t) - x_0 - \int_{t_0}^{t} f(s,x(s))\,ds\| \le (l+1)h\epsilon + \|r(h)\| \le \alpha\epsilon + \|r(h)\| \quad (1.56)$$

which converges to $\alpha\epsilon$ as $h \to 0$. Since ϵ is arbitrary $x(t)$ is a solution of the Cauchy problem in integral form (1.32). ■

1.5 Continuation of solutions

So far we only considered local solutions, i.e., solutions which are defined in a neighborhood of (t_0, x_0). Simple examples shows that the solution $x(t)$ may not exist for all t, for example the equation $x' = 1 + x^2$ has solution $x(t) = \tan(t-c)$ and this solution does not exist beyond the interval $(c-\pi/2, c+\pi/2)$, and we have $x(t) \to \pm\infty$ as $t \to c \pm \pi/2$.

To extend the solution we solve the Cauchy problem locally, say from t_0 to $t_0+\alpha$ and then we can try to continue the solution by solving the Cauchy problem $x' = f(t,x)$ with new initial condition $x(t_0+\alpha)$ and find a solution from $t_0+\alpha$ to $t_0+\alpha+\alpha'$, and so on... In order do this we should be able to solve it locally everywhere and we will therefore assume that f satisfy a *local Lipschitz condition*.

Definition 1.5.1 A function $f : U \to \mathbf{R}^n$ (where U is an open set of $\mathbf{R}\times\mathbf{R}^n$) satisfies a *local Lipschitz condition* if for any $(t_0, x_0) \in U$ there exist a neighborhood $V \subset U$ such that f satisfies a Lipschitz condition on V, see (1.39).

Note that if the function f is of class $\mathcal{C}^1$ in U, then it satisfies a local Lipschitz condition.

Lemma 1.5.2 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and let us assume that $f : U \to \mathbf{R}^n$ is continuous and satisfies a local Lipschitz condition. Then for any $(t_0, x_0) \in U$ there exists an open interval $I_{\max} = (\omega_- , \omega_+)$ with $-\infty \leq \omega_- < t_0 < \omega_+ \leq \infty$ such that*

- *The Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ has a unique solution on $I_{\max}$.*
- *If $y : I \to \mathbf{R}^n$ is a solution of $x' = f(t, x)$, $y(t_0) = x_0$, then $I \subset I_{\max}$ and $y = x|_I$.*

Proof: a) Let $x : I \to \mathbf{R}^n$ and $z : J \to \mathbf{R}^n$ be two solutions of the Cauchy problem with $t_0 \in I, J$. Then $x(t) = z(t)$ on $I \cap J$. Suppose it is not true, there is point $\bar{t}$ such that $x(\bar{t}) \neq z(\bar{t})$. Consider the first point where the solutions separate. The local existence theorem 1.3.4 shows that it is impossible.

b) Let us define the interval

$$I_{\max} = \bigcup \{I\,;\, I \text{ open interval },\ t_0 \in I\,,\ \text{there exists a solution on } I\}\ . \tag{1.57}$$

This interval is open and we can define the solution on $I_{\max}$ as follows. If $t \in I_{\max}$, then there exists I where the Cauchy problem has a solution and we can define $x(t)$. The part (a) shows that $x(t)$ is uniquely defined on $I_{\max}$. ∎

Theorem 1.5.3 *Let $U \subset \mathbf{R}\times\mathbf{R}^n$ be an open set and let us assume that $f : U \to \mathbf{R}^n$ is continuous and satisfies a local Lipschitz condition. Then every solution of $x' = f(t, x)$ has a continuation up to the boundary of U. More precisely, if $x : I_{\max} \to \mathbf{R}^n$ is the solution passing through $(t_0, x_0) \in U$, then for any compact $K \subset U$ there exists $t_1, t_2 \in I_{\max}$ with $t_1 < t_0 < t_2$ such that $(t_1, x(t_1)) \notin K$, $(t_2, x(t_2)) \notin K$.*

Remark 1.5.4 If $U = \mathbf{R} \times \mathbf{R}^n$, Theorem 1.5.3 means that either

- $x(t)$ exists for all t,
- There exists t^* such that $\lim_{t\to t^*} \|x(t)\| = \infty$,

The exists globally or the solution "blows up" at a certain point in time.

Proof: Let $I_{\max} = (\omega_- \,,\, \omega_+)$. If $\omega_+ = \infty$, clearly there exists a point t_2 such that $t_2 > t_0$ and $(t_2, x(t_2)) \notin K$ because K is bounded. If $\omega_+ < \infty$, let us assume that there exist a compact K such that $(t, x(t)) \in K$ for any $t \in (t_0, \omega_+)$. Since $f(t,x)$ is bounded on the compact set K, we have, for t, t' sufficiently close to ω_+

$$\|x(t) - x(t')\| \;=\; \left\| \int_{t'}^{t} f(s, x(s))\, dt \right\| \le M|t - t'| < \epsilon \,. \tag{1.58}$$

This shows that $\lim_{t \to \omega_+} x(t) = x_+$ exists and $(\omega_+, x_+) \in K$, since K is closed. Theorem 1.3.4 for the Cauchy problem with $x(\omega_+) = x_+$ implies that there exists a solution in a neighborhood of ω_+. This contradicts the maximality of the interval $I_{\max}$. For t_1 the argument is similar. ∎

1.6 Global existence

In this section we derive sufficient conditions for *global existence* of solutions, i.e., absence of blow-up for $t > t_0$ or for all t. The following simple lemma and its variants will be very useful.

Lemma 1.6.1 (Gronwall Lemma) *Suppose that $g(t)$ is a continuous function with $g(t) \ge 0$ and that there exits constants $a, b > 0$ such that*

$$g(t) \;\le\; a + b \int_{t_0}^{t} g(s)\, ds \,, \quad t \in [t_0, T] \,. \tag{1.59}$$

Then we have

$$g(t) \le a e^{b(t - t_0)} \quad t \in [t_0, T] \,. \tag{1.60}$$

Proof: Set $G(t) = a + b \int_0^t g(s)\, ds$. Then $G(t) \ge g(t)$, $G(t) > 0$, for $t \in [t_0, T]$, and $G'(t) = bg(t)$. Therefore

$$\frac{G'(t)}{G(t)} \;=\; \frac{bg(t)}{G(t)} \;\le\; \frac{bG(t)}{G(t)} \;=\; b \,, \quad t \in [t_0, T] \,, \tag{1.61}$$

or, equivalently,

$$\frac{d}{dt} \log G(t) \le b \,, \quad t \in [t_0, T] \,, \tag{1.62}$$

or

$$\log G(t) - logG(0) \le b(t - t_0) \,, \quad t \in [t_0, T] \,, \tag{1.63}$$

or

$$G(t) \le G(0) e^{b(t - t_0)} \;=\; a e^{b(t - t_0)} \,, \quad t \in [t_0, T] \,, \tag{1.64}$$

which implies that $g(t) \le a e^{b(t - t_0)}$, for $t \in [t_0, T]$. ∎

The first condition for global existence is rather restrictive, but it has the advantage of being easy to check.

Definition 1.6.2 We say that the function $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is *linearly bounded* if there exists a constant C such that

$$\|f(t,x)\| \leq C(1+\|x\|)\,, \qquad \text{for all } (t,x) \in \mathbf{R} \times \mathbf{R}^n\,. \tag{1.65}$$

Obviously if $f(t,x)$ is bounded on $\mathbf{R}\times\mathbf{R}^n$, then it is linearly bounded. The functions $x\cos(x^2)$, or $x/\log(2+|x|)$ are examples of linearly bounded function. The function $f(x,y) = (x+xy, y^2)^T$ is not linearly bounded.

Theorem 1.6.3 *Let $f : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ be continuous, locally Lipschitz (see Definition 1.5.1) and linearly bounded (see Definition 1.6.2). Then the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, has a unique solution for all t.*

Proof: . Since f is locally Lipschitz, there is a unique local solution $x(t)$. We have the a-priori bound on solutions

$$\|x(t)\| \;\leq\; \|x_0\| + \int_{t_0}^{t} \|f(s,x(s))\|\,ds \leq \|x_0\| + C\int_{t_0}^{t} (1+\|x(s)\|)\,ds\,, \tag{1.66}$$

Using Gronwall Lemma for $g(t) = 1 + \|x(t)\|$, we find that

$$1+\|x(t)\| \leq (1+\|x_0\|)e^{C(t-t_0)}\,, \quad \text{or} \quad \|x(t)\| \leq \|x(0)\|e^{C(t-t_0)} + (e^{C(t-t_0)}-1)\,. \tag{1.67}$$

This shows that the norm of the solution grows at most exponentially fast in time. From Remark 1.5.4 it follows that the solution does not blow up in finite time. ■

We formulate additional sufficient conditions for global existence but, for simplicity, we restrict ourselves to *autonomous* equations: we consider Cauchy problems of the form

$$x' = f(x)\,, \qquad x(t_0) = x_0\,, \tag{1.68}$$

where $f(t,x)$ does not depend explicitly on t.

Theorem 1.6.4 (Liapunov functions) *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be locally Lipschitz. Suppose that there exists a function $V(x) : \mathbf{R}^n \to \mathbf{R}$ of class $\mathcal{C}^1$ such that*

- $V(x) \geq 0$ *and* $\lim_{\|x\|\to\infty} V(x) = \infty$.
- $\langle \nabla V(x)\,,\, f(x)\rangle \leq a + bV(x)$

Then the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$, has a unique solution for $t_0 < t < +\infty$.

Proof: Since f is locally Lipschitz, there is a unique local solution $x(t)$ for the Cauchy problem. We have

$$\frac{d}{dt}V(x(t)) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\frac{dx_j}{dt} = \langle \nabla V(x(t)) \,,\, f(x(t))\rangle \leq a + bV(x(t) \tag{1.69}$$

or, by integrating,

$$V(x(t)) \leq V(x(t_0)) + \int_{t_0}^{t} (a + bV(x(s))\, ds\,. \tag{1.70}$$

Applying Gronwall lemma to $g(t) = a + bV(x(t))$ gives the bound

$$a + bV(x(t)) \leq (a + bV(x(0)))\, e^{b(t-t_0)}\,. \tag{1.71}$$

Therefore $V(x(t))$ remains bounded for all t. Since $\lim_{\|x\|\to\infty} V(x) = \infty$, the level sets of V, $V^{-1}(c)$ are compact for all c and thus $\|x(t)\|$ stays finite for all $t > t_0$ too. ∎

Remark 1.6.5 The function V in Theorem 1.6.4 is usually referred to as a *Liapunov function*. We will also use similar function later to study the stability of solutions. Note that there is no general method to construct Liapunov function, it involves some trial and error and some a-priori knowledge on the equation.

Example 1.6.6 (Gradient systems) Let $V : \mathbf{R}^n \to \mathbf{R}$ be a function of class $\mathcal{C}^2$. A *gradient systems* is an ODE of the form

$$x' = -\nabla V(x)\,. \tag{1.72}$$

(The negative sign is a traditional convention). Note that in dimension $n = 1$, any autonomous ODE $x' = f(x)$ is a gradient system since we can always write $V(x) = \int_{x_0}^{x} f(y)\, dy$.

Consider the level sets of the function V, $V^{-1}(c) = \{x\,; V(x) = c\}$. If $x \in V^{-1}(c)$ is a *regular point*, i.e., if $\nabla V(x) \neq 0$, then, by the implicit function Theorem, locally near x, $V^{-1}(c)$ is a smooth hypersurface surface of dimension $n-1$. For example, if $n = 2$, the level sets are smooth curves.

Note that if x is a regular point of the level curve $V^{-1}(c)$, then the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$. Indeed let y be a vector which is tangent to the level surface $V^{-1}(c)$ at the point x. For any curve $\gamma(t)$ in the level set $V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$ we have

$$0 = \frac{d}{dt}V(\gamma(t))|_{t=0} = \langle \nabla V(x) \,,\, y\rangle\,, \tag{1.73}$$

and so $\nabla V(x)$ is perpendicular to any tangent vector to the level set $V^{-1}(c)$ at all regular points of V.

We have the following

Lemma 1.6.7 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be a function of class $\mathcal{C}^2$ with $\lim_{\|x\|\to\infty} V(x) = +\infty$. Then any solution of the gradient system $x' = -\nabla V(x)$, $x(t_0) = x_0$ exists for all $t > t_0$.*

Proof: If $x(t)$ is a solution of (3.107), then we have

$$\frac{d}{dt}V(x(t)) \,=\, -\langle \nabla V(x(t))\,,\, \nabla V(x(t))\rangle \,\le\, 0\,. \tag{1.74}$$

This shows that V is a Liapunov function. ■

Example 1.6.8 (Hamiltonian systems.) Let $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, and $H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a function of class $\mathcal{C}^2$. The function $H(x,y)$ is called a *Hamiltonian function* (or *energy function*) and the $2n$-dimensional ODE

$$x' \,=\, \nabla_y H(x,y)\,, \quad y' \,=\, -\nabla_x H(x,y)\,. \tag{1.75}$$

is called the *Hamiltonian equation* for the Hamiltonian $H(x,y)$. Since H is of class $\mathcal{C}^2$, the vector field $f(x,y) = (\nabla_y H(x,y)\,,\, -\nabla_y H(x,y))^T$ is locally Lipschitz so that we have local solutions. Let $(x(t), y(t))$ be a solution of (1.75). We have then

$$\begin{aligned}\frac{d}{dt}H(x(t),y(t)) \;&=\; \nabla_x H \cdot x' + \nabla_y H \cdot y'(t) \\ &=\; \nabla_x H \cdot \nabla_y H - \nabla_y H \cdot \nabla_x H \,=\, 0\,.\end{aligned} \tag{1.76}$$

This means that H is a *integral of the motion*, for any solution $H(p(t), q(t)) = \text{const}$ and that any solution stays on a level set of the function H. For Hamiltonian equations this usually referred to as conservation of energy.

Let us assume further that $\lim_{\|(x,y)\|\to\infty} H(x,y) = \infty$. This means that $H(x,y)$ is bounded below, i.e., $H(x,y) \ge -c$ from some $c \in \mathbf{R}$ and that the level sets $\{H(x,y) = c\}$ are closed and bounded hypersurfaces. In this case $H(x,y)+c$ is a Liapunov function for the ODE (1.75) and the solution exist for all positive and negative times.

Example 1.6.9 (van der Pol equations) The second order equation $x'' = \epsilon(1 - x^2)x' - x$ is written as the first order system

$$\begin{aligned} x' \;&=\; y \\ y' \;&=\; \epsilon(1-x^2)y - x \end{aligned} \tag{1.77}$$

and is a perturbation of the harmonic oscillator $x''+x = 0$ which is an Hamiltonian system with Hamiltonian $H(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$ (harmonic oscillator). Taking the Hamiltonian as the Liapunov function we have

$$\langle \nabla H(y,x)\,,\, f(y,x)\rangle \,=\, \epsilon(1-x^2)y^2 \,=\, \begin{cases} \le 0 & \text{if} \quad x^2 \ge 1 \\ \le \epsilon y^2 & \text{if} \quad x^2 \le 1 \end{cases}\,. \tag{1.78}$$

Therefore $\nabla H \cdot f \le 2\epsilon H$ and H is a Liapunov function and we obtain global existence of solutions.

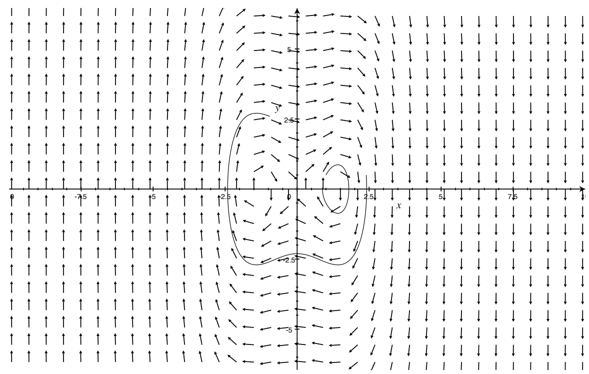

Figure 1.6: The vector field for the Hamiltonian $x^4 - 4x^2 + y^2$ and two solutions between $t = 0$ and $t = 3$ with initial conditions $(0.5, 1)$ and $(0.5, 2.4)$

Another class of systems which have solutions for all times are given by *dissipative systems*.

Theorem 1.6.10 (Dissipative systems) *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be locally Lipschitz. Suppose that there exists $v \in \mathbf{R}^n$ and positive constants a and b such that*

$$\langle f(x)\,,\, x - v\rangle \,\leq\, a - b\|x\|^2\,. \tag{1.79}$$

Then the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$, has a unique solution for $t_0 < t < +\infty$.

Proof: Consider the balls $B_0 = \{x \in \mathbf{R}^n\,;\, \|x\|^2 \leq a/b\}$ and the Liapunov function

$$V(x) \,=\, \frac{||x - v||^2}{2}\,. \tag{1.80}$$

The condition (1.79) implies that for any solution $\frac{d}{dt}V(x(t)) \leq 0$ outside of the ball B_0 and therefore V is a Liapunov function. ■

Remark 1.6.11 The condition (1.79) means that for the balls $B = \{x \in \mathbf{R}^n\,;\, \|x - v\|^2 \leq R\}$ with $R \geq \|v\| + \sqrt{a/b}$ is chosen so large that B_0 is contained in the interior of B, the vector field f points toward the interior of B. This implies that a solution which starts in B will stay in B forever.

There are many variants to Theorem 1.6.10 (see the exxercises). The basic idea is to find a family of sets (large balls in Theorem 1.6.10 but the set could have other shapes) such that, on the boundary of the sets the vector f points inward. This implies that solutions starting on the boundary will move inward the set. If one proves this for all sufficiently large sets, then one obtains global existence for all initial data.

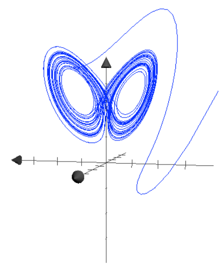

Figure 1.7: The solution for Lorentz equation with $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$ and initial condition (-40, 40, 25)

Example 1.6.12 The Lorentz equations are given by

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= -x_1 x_3 + r x_1 - x_2 \\ x_3' &= x_1 x_2 - b x_3 \end{aligned} \tag{1.81}$$

Despite its apparent simplicity, the Lorentz equations exhibits, for suitable values of the parameters, a very complex behavior. All solutions are attracted to a compact invariant set on which the motion is chaotic. Such an invariant set is called a strange attractor (see Figure 1.7).

We show that the system is dissipative, we take $v = (0, 0, \gamma)$. Choosing $\gamma = b + r$ and using the inequality $2\gamma x_3 \leq \gamma^2 + x_3^2$, we find

$$\begin{aligned} \langle f(x)\,,\, x - v\rangle &= -\sigma x_1^2 - x_2^2 - b y_3^2 + (\sigma + r - \gamma) x_1 x_2 + b\gamma x_3 \\ &= -\sigma x_1^2 - x_2^2 - b y_3^2 + b\gamma x_3 \\ &\leq -\sigma x_1^2 - x_2^2 - \frac{b}{2} y_3^2 + b\frac{\gamma^2}{2}\,. \end{aligned} \tag{1.82}$$

If $\gamma = b + r$, then (1.79) is satisfied with $\alpha = b\frac{\gamma^2}{2}$ and $\beta = \min(\sigma, 1, b/2)$ and the solution of Lorentz systems exists for all $t > 0$.

1.7 Wellposedness and dynamical systems

For the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$, we denote the solution by $x(t, t_0, x_0)$ where we explicitly indicate the dependence on the initial time t_0 and the initial position x_0.

Definition 1.7.1 The Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ is called *locally wellposed* (resp. *globally wellposed*) if there exists a unique local (resp. global) solution $x(t, t_0, x_0)$ which depends continuously of (t_0, x_0).

Lemma 1.7.2 *Let $f : U \to \mathbf{R} \times \mathbf{R}^n$ (U an open set of $\mathbf{R} \times \mathbf{R}^n$) be continuous and satisfy a local Lipschitz condition. Then for any compact $K \subset U$ there exists $L \geq 0$ such that*

$$\|f(t, y) - f(t, x)\| \leq L\|x - y\|\,, \qquad \text{for all } (t, x), (t, y) \in K \tag{1.83}$$

Proof: Let us assume the contrary. Then there exists sequences (t_n, x_n) and (t_n, y_n) in K such that

$$\|f(t_n, x_n) - f(t_n, y_n)\| > n\|x_n - y_n\|\,. \tag{1.84}$$

Since f is bounded on K with $M = \max_{(t,x)\in K} \|f(t, x)\|$, it follows from (1.84) that

$$\|x_n - y_n\| \leq 2M/n\,. \tag{1.85}$$

By Bolzano-Weierstrass, the sequence (t_n, x_n) has an accumulation point (t, x), and $f(t, x)$ satisfies a Lipschitz condition in a neighborhood V of (t, x).

The bound (1.85) implies that there are infinitely many indices n such that $(t_n, x_n) \in V$ and $(t_n, y_n) \in V$. Then (1.84) contradicts the Lipschitz condition on V. ∎

Theorem 1.7.3 *Let $f : U \to \mathbf{R} \times \mathbf{R}^n$ (U an open set of $\mathbf{R} \times \mathbf{R}^n$) be continuous and satisfy a local Lipschitz condition. Then the solution $x(t, t_0, x_0)$ of the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ is a continuous function of (t_0, x_0). Moreover the function $x(t, t_0, x_0)$ is a Lipschitz continuous function of x_0, i.e., there exists a constant $R = R(t)$ such that*

$$\|x(t, t_0, x_0) - x(t, t_0, x_1)\| \leq R\|x_0 - x_1\|\,. \tag{1.86}$$

Proof: We choose a closed subinterval $[a, b]$ of the maximal interval of existence $I_{\max}$ with $t, t_0 \in [a, b]$. We choose ϵ small enough such that the tubular neighborhood K around the solution $x(t, t_0, x_0)$,

$$K = \{(t, x)\,;\, t \in [a, b]\,,\, \|x - x(t, t_0, x_0)\| \leq \epsilon\}\,, \tag{1.87}$$

is contained in the open set U. By Lemma 1.7.2, $f(t, x)$ satisfies a Lipschitz condition on K with a Lipschitz constant L. The set V

$$V = \left\{(t_1, x_1)\,;\, t_1 \in [a, b]\ \ \|x_1 - x(t_1, t_0, x_0)\| \leq \epsilon e^{-L(b-a)}\right\}\,, \tag{1.88}$$

is a neighborhood of (t_0, x_0) which satisfies $V \subset K \subset U$. If $(t_1, x_1) \in V$ we have

$$\begin{aligned} \|x(t, t_1, x_1) - x(t, t_0, x_0)\| &= \|x(t, t_1, x_1) - x(t, t_1, x(t_1, t_0, x_0))\| \\ &\leq \|x_1 - x(t_1, t_0, x_0)\| + L\int_{t_1}^{t} \|x(s, t_1, x_1) - x(s, t_1, x(t_1, t_0, x_0))\|\,. \end{aligned} \tag{1.89}$$

From Gronwall lemma we conclude that

$$\|x(t,t_1,x_1) - x(t,t_0,x_0)\| \leq e^{L|t-t_1|}\|x_1 - x(t_1,t_0,x_0)\| \leq \epsilon \tag{1.90}$$

and this concludes the continuity of $x(t,t_0,x_0)$. To prove the Lipschitz continuity in x_0 one sets $t_1 = t_0$ in (1.90) and this prove (1.86) with $R = e^{L|t-t_0|}$. ∎

This theorem shows that the Cauchy problem is locally wellposed, provided f is continuous and satisfy a local Lipschitz condition. If, in addition, the solutions exist for all times then the Cauchy problem is globally wellposed.

Let us consider the map $\phi^{t,t_0} : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\phi^{t,t_0}(x_0) = x(t,t_0,x_0)\,. \tag{1.91}$$

The map ϕ^{t,t_0} maps the initial position x_0 at time t_0 to the position at time t, $x(t,t_0,x_0)$. By definition the maps ϕ^{t,t_0} satisfy the composition relations

$$\phi^{t+s,t_0}(x_0) = \phi^{t+s,t}(\phi^{t,t_0}(x_0))\,. \tag{1.92}$$

If the ODE is autonomous, i.e., $f(x)$ does not depend explicitly on t we have

Lemma 1.7.4 (Translation property) *Suppose that $x(t)$ is a solution of $x' = f(x)$, then $x(t-t_0)$ is also a solution.*

Proof: If $x'(t) = f(x(t))$, then $\frac{d}{dt}x(t-t_0) = x'(t-t_0) = f(x(t-t_0))$. ∎

This implies that, if $x(t) = x(t,0,x_0)$ is the solution of the Cauchy problem $x' = f(x)$, $x(0) = x_0$, then $x(t-t_0)$ is the solution of the Cauchy problem $x' = f(x)$, $x(t_0) = x_0$. In other words $x(t-t_0) = x(t,t_0,x_0)$ and so the solution depends only on $t - t_0$. For autonomous equations we can thus always assume that $t_0 = 0$. In this case we will denote then the map $\phi^{t,t_0} = \phi^{t-t_0,0}$ simply by ϕ^{t-t_0}. The map ϕ^t has the following group properties

- **(a)** $\phi^0(x) = x$.
- **(b)** $\phi^t(\phi^s(x)) = \phi^{t+s}(x)$.
- **(c)** $\phi^t(\phi^{-t}(x)) = \phi^{-t}(\phi^t(x)) = x$.

If the solutions exists for all $t \in \mathbf{R}$, the collection of maps ϕ^t is called the *flow* of the differential equations $x' = f(x)$. Note that Property (c) implies that the map $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$ is invertible. More generally, a continuous map $\phi^{\cdot} : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ which satisfies Properties (a)-(b)-(c) is called a (continuous time) *dynamical system.* If the Cauchy problem is globally wellposed then the maps ϕ^t are a continuous flow of homeomorphisms and we will say that the dynamical system is continuous.

Remark 1.7.5 If the vector fields $f(t,x)$ are of class $\mathcal{C}^k$ then one would expect that $x(t,t_0,x_0)$ is also a function of class $\mathcal{C}^k$. We will discuss this in the next chapter.

Remark 1.7.6 Theorem 1.7.3 shows the following: For fixed t, $x(t,t_0,x_0)$ can be made arbitrarily close to $x(t,t_0,x_0+\xi)$ provided ξ is small enough (depending on t!). This does not mean however that the solutions which start close to each other will remain close to each other, What we proved is a bound $\|x(t,t_0,x_0+\xi)-x(t,t_0,x_0)\| \le K\|\xi\|e^{L|t-t_0|}$ which show that two solutions can separate, typically at an exponential rate.

Example 1.7.7 For the Cauchy problem

$$x' = \begin{pmatrix} -1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.93}$$

the solution is $(e^{-t},0)^T$. The solution with initial condition $(1,\xi)^T$ is $(e^{-t},\xi e^{\kappa t})^T$. If $\kappa \le 0$ both solutions stay a distance less than $|\xi|$ for all time $t \ge 0$, if $\kappa > 0$ the solutions diverge from each other exponentially with time. For given t, we can however make them arbitrarily close up to time t by choosing ξ small enough, hence the continuity.

1.8 Exercises

1. Determine whether the following sequences of functions are Cauchy sequences with respect to the uniform norm $\|\cdot\|_\infty$ on the given interval I. Determine the limit $f_n(x)$ if it exists.
   (a) $f_n(x) = \sin(2\pi n x)\,, \quad I = [0,1]$
   (b) $f_n(x) = \dfrac{x^n-1}{x^n+1}\,, \quad I = [-1,1].$
   (c) $f_n(x) = \dfrac{1}{n^2+x^2}\,, \quad I = [0,1]$
   (d) $f_n(x) = \dfrac{nx}{1+(nx)^2}\,, \quad I = [0,1]$

2. Show that $\|f\|_2$ is a norm on $\mathcal{C}([0,1])$.

3. Prove that all norms on $\mathbf{R}^n$ are equivalent by proving that any norm $\|\cdot\|$ in $\mathbf{R}^n$ is equivalent to the euclidean norm $\|\cdot\|_2$.
   Hint: (a) Let e_i be the usual vector basis in $\mathbf{R}^n$ and write $x = x_1e_1 + \cdots + x_ne_n$ and use the triangle inequality and Cauchy-Schwartz to show that $\|x\| \le C\|x\|_2$.
   (b) Using (a) prove that $|\,\|x\| - \|y\|\,| \le C\|x-y\|_2$ and thus the function $\|\cdot\|$ on $\mathbf{R}^n$ with $\|\|_2$ is a continuous function.
   (c) Consider now the function $\|\cdot\|$ on the compact set $K = \{x\,,\ \|x\|_2 = 1\}$ and deduce from this the equivalence of the two norms.

4. (a) Let $f : U \to \mathbf{R}^n$ where $U \subset \mathbf{R}^n$ is an open set and suppose that f satisfies a Lipschitz condition on U. Show that f is uniformly continuous on U.

   (b) Let $f : E \to \mathbf{R}^n$ where $E \subset \mathbf{R}^n$ is a compact set. Suppose that f is locally Lipschitz on E, show that f satisfies a Lipschitz condition on E.

   (c) Show that $f(x) = 1/x$ is locally Lipschitz but that it does not satisfy a Lipschitz condition on $(0, 1)$.

   (d) Show that $f(x) = \sqrt{|x|}$ is not locally Lipschitz.

   (e) Does the Cauchy problem $x' = 1/x$, $x(0) = a > 0$ have a unique solution? Solve it and determine the maximal interval of existence. What is the behavior of the solution at the boundary of this interval?

5. (a) Derive the following *error estimate* for the method of successive approximations. Let x be a fixed point given by this method. Show that

$$\|x - x_k\| \leq \frac{\alpha}{1-\alpha}\|x_k - x_{k-1}\|\,, \tag{1.94}$$

   where α is the contraction rate.

   (b) Consider the function $f(x) = e^x/4$ on the interval $[0, 1]$. Show that f has a fixed point on $[0, 1]$. Do some iterations and estimate the error rigorously using (a).

6. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x + e^{-x/2} & \text{if } x \geq 0 \\ e^{x/2} & \text{if } x \leq 0 \end{cases}\,. \tag{1.95}$$

   (a) Show that $|f(x) - f(y)| < |x - y|$ for $x \neq y$.

   (b) Show that f does not have a fixed point.

   Explain why this does not contradict the Banach fixed point theorem.

7. Consider the IVP

$$x' = x^3\,, \quad x(0) = a\,. \tag{1.96}$$

   (a) Apply the Picard-Lindelöf iteration to compute the first three iterations $x_1(t)$, $x_2(t)$, $x_3(t)$.

   (b) Find the exact solution and expand it in a Taylor series around $t = 0$. Show that the first few terms agrees with the Picard iterates.

   (c) How does the number of correct terms grow with iteration?

8. Apply the Picard-Lindelöf iteration to the Cauchy problem

$$x_1' = x_1 + 2x_2\,, \qquad x_1(0) = 0 \tag{1.97}$$
$$x_2' = t^2 + x_1\,, \qquad x_2(0) = 0 \tag{1.98}$$

Compute the first five terms in the taylor series of the solution.

9. Show that the assumption that "D is closed" cannot be omitted in general in the fixed point theorem. Find a set D which is not closed and a map $f : D \to E$ such that $f(D) \subset D$, f is a contraction, but f does not have a fixed point in D.

10. (a) Let $I = [t_0 - \alpha, t_0 + \alpha]$ and for a positive constant κ define

$$\|x\|_\kappa = \sup_{t \in I} \|x(t)\| e^{-\kappa|t-t_0|}\,.$$

Show that $\|\cdot\|_\kappa$ defines a norm and that the space

$$E = \{x : I \to \mathbf{R}^n\,,\ x(t) \text{ continuous and } \|\text{x}\|_\kappa < \infty\}$$

is a Banach space.

(b) Consider the IVP $x' = f(t, x)$, $x(t_0) = x_)$. Give a proof of Theorem 1.3.4 in the classnotes by applying the Banach fixed point theorem in the Banach space E with norm $\|\cdot\|_\kappa$ for a well-chosen κ.

(c) Suppose that $f(t, x)$ satisfy a *global Lipschitz condition*, i.e., there exists a positive $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbf{R}^n \text{ and for all } t \in \mathbf{R}\,. \tag{1.99}$$

Show that the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ has a unique solution for all $t \in \mathbf{R}$. *Hint:* Use the norm defined in (a).

11. Consider the map T given by

$$T(f)(x) = \sin(2\pi x) + \lambda \int_{-1}^{1} \frac{f(y)}{1 + (x-y)^2}\, dy$$

(a) Show that if $f \in \mathcal{C}([-1, 1], \mathbf{R})$ then so is $T(f)$.

(b) Find a λ_0 such that T is a contraction if $|\lambda| < \lambda_0$ and T is not a contraction if $|\lambda| > \lambda_0$. *Hint:* For the second part find a pair f, g such that $\|T(f) - T(g)\|_\infty > \|f - g\|_\infty$.

12. (a) Consider the norm of $\mathcal{C}([0,a])$ given by
$$\|f\|_{\mathrm{e}} \,=\, \max_{0\le t\le a} |f(t)|e^{-t^2}\,. \tag{1.100}$$
(Why is it a norm?) Let
$$Tf(t) \,=\, \int_0^t sf(s)\,ds\,. \tag{1.101}$$
Show that $\|Tf\|_\infty \le \frac{a^2}{2}\|f\|_\infty$ and $\|Tf\|_{\mathrm{e}} \le \frac{1}{2}\|f\|_{\mathrm{e}}$.

(b) Show that the integral equation
$$x(t) \,=\, \frac{1}{2}t^2 + \int_0^t sx(s)\,ds\,, \quad t\in[0,a]\,, \tag{1.102}$$
has exactly one solution. Determine the solution (i) by rewriting the equation as an initial value problem and solving it, (ii) by using the methods of successive approximations starting with $x_0 \equiv 0$.

13. Let us consider $\mathbf{R}^2$ with the norm $\|x\| = \max\{|x_1|,|x_2|\}$. Let $f\,;\,\mathbf{R}^2 \to \mathbf{R}^2$ be given by
$$f(x_1,x_2) \,=\, \begin{pmatrix} x_1^2 + 2x_2^2 + 5\cos(x_2) \\ 4x_1x_2 + 3 \end{pmatrix} \tag{1.103}$$
Let $K = \{(x_1,x_2)\,,\,|x_1|<1, |x_2|\le 2\}$. Find an explicit Lipschitz constant L for f on K.

14. Let $f\,:\,\mathbf{R}^2 \to \mathbf{R}$ be of class $\mathcal{C}^1$ and satisfy $f(0,0)=0$. Suppose that $x(t)$ is a solution of the ODE
$$x'' \,=\, f(x,x')\,, \tag{1.104}$$
which is not identically 0. Show that $x(t)$ has simple zeros. Examples: the harmonic oscillator $x''+x=$ or the mathematical pendulum $x''+\sin(x)=0$.

15. Consider the initial value problem $x' = f(t,x)$, $x(t_0)=x_0$, where $f(t,x)$ is a continuous function. Show that if the initial value problem has a unique solution then the Euler polygons $x_h(t)$ converge to this solution.

16. Consider the Cauchy problem $x' = f(t,x)$, $x(0)=0$, where
$$f(t,x) \,=\, \begin{cases} 4\mathrm{sign}(x)\sqrt{|x|} & \text{if} \quad |x|\ge t^2 \\ 4\mathrm{sign}(x)\sqrt{|x|} + 4(t-\frac{|x|}{t})\cos(\pi\frac{\log t}{\log 2}) & \text{if} \quad |x|<t^2 \end{cases} \tag{1.105}$$
The function f is continuous on $\mathbf{R}^2$. Consider the Euler polygons $x_h(t)$ with $h=2^{-i}$, $i=1,2,3,\cdots$. Show that $x_h(t)$ does not converge as $h\to 0$, compute its accumulation points, and show that they are solution of the Cauchy problem. *Hint: the solutions are* $\pm 4t^2$.

17. Consider the the Cauchy problem $x' = f(t,x)$, $x(0) = 0$ where f is given by

$$f(t,x) = \begin{cases} 0 & \text{if } t \le 0, \quad x \in \mathbf{R} \\ 2t & \text{if } t > 0, \quad x \le 0 \\ 2t - \frac{4x}{t} & \text{if } t > 0, \quad 0 \le x < t^2 \\ -2t & \text{if } t > 0, \quad t^2 \le x \end{cases} \tag{1.106}$$

    (a) Show that f is continuous. What does that imply for the Cauchy problem?

    (b) Show that f does not satisfy a Lipschitz condition in any neighborhood of the origin.

    (c) Apply Picard-Lindelöf iteration with $x_0(t) \equiv 0$. Are the accumulation points solutions?

    (d) Show that the Cauchy problem has a unique solution. What is the solution?

    This problem shows that existence and uniqueness of the solution does not imply that the Picard-Lindelöf iteration converges to the unique solution.

18. Consider the Cauchy problem $x' = \lambda x$, $x(0) = 1$, with $\lambda > 0$ and $t \in [0,1]$. Compute the Euler polygons $x_h(t)$ with $h = 1/n$ and show that

$$\frac{\lambda}{1+\lambda h} x_h(t) \le \frac{dx_h}{dt}(t) \le \lambda x_h(t)\,. \tag{1.107}$$

    Deduce from this the classical inequality

$$\left(1 + \frac{\lambda}{n}\right)^n \le e^\lambda \le \left(1 + \frac{\lambda}{n}\right)^{n+\lambda} \tag{1.108}$$

    Hint: Use Gronwall Lemma.

19. Let a, b, c, and d be positive constants. Consider the Predator-Prey equation $x' = x(a - by)$, $y' = y(cx - d)$ with positive initial conditions $x(t_0) > 0$ and $y(t_0) > 0$. Show that the solutions exists for all t and that the solution curves $x(t), y(t)$ are periodic. *Hint:* You can use the change of variables $p = \log(x)$ and $q = \log(y)$

20. (a) Show that any second order ODE $x'' + f(x) = 0$ can be written as a Hamiltonian system for the Hamiltonian function $H(x,y) = y^2/2 + V(x)$, where $y = x'$ and $V(x) = \int_0^x f(t)dt$

    (b) Compute the Hamiltonian function, and it level curves and draw the solutions curves for the following ODE's

        i. $x'' = -\omega^2 x$ (the harmonic oscillator)

ii. $x'' = -a\sin(x)$ (the mathematical pendulum: One end A of weightless rod of length l is attached to a pivot, and a mass m is attached to the other end B. The system moves in a plane under the influence of the gravitational force of amplitude mg which acts vertically downward. Here $x(t)$ is the angle between the vertical and the rod and $a = g/l$).

iii. $mr'' = -\gamma Mm/r^2$ (Vertical motion of a body of mass m in free fall due to the gravity of a body of mass M).

Depending on the energy $H(x_0, y_0)$ of the initial condition discuss in details the different types of solutions which can occur. Are the solutions bounded or unbounded? Are there constant solutions or periodic solutions? Do the solutions converge as $t \to \pm\infty$?

21. (a) Consider the Hamiltonian function $H(x, y) = y^2/2 + V(x)$. Suppose that we have initial conditions $x(0) = x_0$ and $x'(0) = y_0 > 0$ with initial energy $E = H(x_0, y_0)$. Use the conservation of energy to show the solution $x(t)$ is given (implicitly) by the formula

$$t = \int_{x_0}^{x(t)} \frac{1}{\sqrt{2(E - V(s))}}\, ds\,.$$

(b) Assume that $V(x) = V(-x)$, i.e., V is an even function. Show that if $x(t)$ is a solution then so are $x(c-t)$ and $-x(t)$. Furthermore show that if $x(c) = 0$ then $x(c+t) = -x(c-t)$ and that if $x'(d) = 0$ then $x(d+t) = x(d-t)$.

(c) Assume that $V(x) = V(-x)$ and consider periodic solutions. We denote by R the largest swing, i.e., the maximal positive value of $x(t)$ along the periodic solution. Using (a) show that the period p of the periodic solution is given by

$$p = 4\int_0^R \frac{1}{\sqrt{2(V(R) - V(s))}}\, ds$$

Hint: Consider the quarter oscillation starting at the point $x(0) = 0$ and $y(0) = y_0 > 0$ and ending at $x(T) = R > 0$ and $y(T) = 0$. Use also the symmetry of V and (b).

(d) Use (c) to show the period for the harmonic oscillator is independent of the energy E.

(e) Use (c) to show that for the mathematical pendulum the period is given by

$$p = \frac{4}{\sqrt{a}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2(u)}}\, du$$

where $k = \sin r/2$. This integral is an elliptic integral of the first type. *Hint:* Use $1 - \cos(\alpha) = \sin^2\frac{\alpha}{2}$ and the substitution $\sin s/2 = k\sin u$.

22. Show that the following ODE's have global solutions (i.e., defined for all $t > t_0$).

   (a) $$\begin{aligned} x' &= 4y^3 + 2x \\ y' &= -4x^3 - 2y - \cos(x) \end{aligned} .$$

   (b) $x'' + x + x^3 = 0.$

   (c) $x'' + x' + x + x^3 = 0.$

   (d) $$\begin{aligned} x' &= \frac{\sin(2t^2 x)x^3}{1+t^2+x^2+y^2} \\ y' &= \frac{x^2 y}{1+x^2+y^2} \end{aligned} .$$

   (e) $$\begin{aligned} x' &= 5x - 2y - y^2 \\ y' &= 2y + 6x + xy - y^3 \end{aligned} .$$

23. Prove the following generalizations of Gronwall Lemma.

   - Let $a > 0$ be a positive constant and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

     $$g(t) \le a + \int_0^t h(s)g(s)\, ds\,. \tag{1.109}$$

     Then, for any $t \in [0, T]$

     $$g(t) \le a e^{\int_0^t h(s)\, ds}\,. \tag{1.110}$$

   - Let $f(t) > 0$ be a positive function and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

     $$g(t) \le f(t) + \int_0^t h(s)g(s)\, ds\,. \tag{1.111}$$

     Then, for any $t \in [0, T]$

     $$g(t) \le f(t) e^{\int_0^t h(s)\, ds}\,. \tag{1.112}$$

24. Consider the FitzHugh-Nagumo equation

   $$\begin{aligned} x_1' &= f_1(x_1, x_2) = g(x_1) - x_2\,, \\ x_2' &= f_2(x_1, x_2) = \sigma x_1 - \gamma x_2\,, \end{aligned} \tag{1.113}$$

   where σ and γ are positive constants and the function g is given by $g(x) = -x(x - 1/2)(x - 1)$.

   (a) In the x_1-x_2 plane draw the graph of the curves $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$.

(b) Consider the rectangles $ABCD$ whose sides are parallel to the X_1 and x_2 axis with two opposite corners located on the $f_2(x_1, x_2) = 0$. Show that if the rectangle is taken sufficiently large, a solution which start inside the rectangle stays inside the rectangle forever. Deduce from this that the equations for any initial conditions x_0 have a unique solutions for all time $t > 0$.

25. Show that the solutions of

$$\begin{aligned} x_1' &= x_1(3 - 4x_1 - 2x_2)\,, \\ x_2' &= x_2(4 - 2x_1 - 3x_2)\,, \end{aligned} \tag{1.114}$$

have a unique solution for all $t \geq 0$, for any initial conditions x_{10} , x_{20} wich are nonnegative. *Hint*: A possibility is to use a similar procedure as in the previous exercise.

26. **Continuous dependence on parameters.** Consider the IVP $x' = f(t, x, \mu)$, $x(t_0) = x_0$ where $f : V \to \mathbf{R}^n$ ($V \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ an open set). We denote by $x(t, \mu)$ the solution of the IVP (we have suppressed the dependence on (t_0, x_0)). Let us assume

   - f is a continuous function on V.
   - $f(t, x, c)$ satisfies a local Lipschitz condition in the following sense: Given $(c_0, t_0, x_0) \in V$ and positive constants a, b, c such that $A \equiv \{(t, x, \mu)\,;\, |t - t_0| \leq a\,,\, \|x - x_0\| \leq b\,,\, \|\mu - \mu_0\| \leq c\} \subset V$ then there exists a constant L such that $\|f(t, x, \mu) - f(t, y, \mu)\| \leq L\|x - y\|$ for all $(t, x, \mu), (t, y, \mu) \in A$.

   Show that $x(t, \mu)$ depends continuously on μ for t in some interval J containing t_0.

Chapter 2

Linear Differential Equations

We denote by $\mathcal{L}(\mathbf{R}^n)$ the set of linear maps $A : \mathbf{R}^n \to \mathbf{R}^n$ which we identify with the set of $n \times n$ matrices $A = (a_{ij})$ with real entries. We write Ax instead of $A(x)$ for the vector with coefficients $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$.

In this chapter we consider *linear differential equations*, i.e., ODE's of the form

$$x' = A(t)x + g(t)\,, \tag{2.1}$$

where $x \in \mathbf{R}^n$, $g : I \to \mathbf{R}^n$, and $A : I \to \mathcal{L}(\mathbf{R}^n)$ with I some interval.

The linear ODE is called *homogeneous* if $g(t) \equiv 0$, and *inhomogeneous* otherwise. If $A(t) = A$ is independent of t and $g \equiv 0$, the linear ODE $x' = Ax$ is called a *system with constant coefficients*.

2.1 General theory

We discuss first general properties of the differential equations $x' = A(t)x + g(t)$.

Theorem 2.1.1 (Existence and uniqueness) *Let $I = [a, b]$ be an interval and suppose that $A(t)$ and $g(t)$ are continuous function on I. Then the Cauchy problem $x' = A(t)x + g(t)$, $x(t_0) = x_0$ (with $t_0 \in I$, $x_0 \in \mathbf{R}^n$) has a unique solution on I.*

Proof: The function $f(t, x) = A(t)x + g(t)$ is continuous and satisfies a Lipschitz condition on $I \times \mathbf{R}^n$. Therefore the solution is unique wherever it exists. Moreover on $I \times \mathbf{R}^n$ we have the bound $\|f(t, x)\| \le a\|x\| + b$ where $a = \sup_{t\in I} \|A(t)\|$ and $b = \sup_{t\in I} \|g(t)\|$. Therefore we have the bound $\|x(t)\| \le \|x_0\| + \int_{t_0}^t (a\|x(s)\| + b)\, ds$ for $t_0, t \in I$. Gronwall lemma implies that $\|x(t)\|$ remains bounded if $t \in I$. ∎

Remark 2.1.2 If $A(t)$ and $g(t)$ are continuous on $\mathbf{R}$ then, applying Theorem 2.1.1 to $[-T, T]$ for arbitrary T shows that the solution exists for all $t \in \mathbf{R}$.

Theorem 2.1.3 (Superposition principle) *Let I be an interval and let $A(t)$, $g_1(t)$, $g_2(t)$ be continuous function on I. If*

$$\begin{aligned} x_1 \,:\, I \to \mathbf{R}^n \quad & \text{is a solution of } x' = A(t)x + g_1(t)\,, \\ x_2 \,:\, I \to \mathbf{R}^n \quad & \text{is a solution of } x' = A(t)x + g_2(t)\,, \end{aligned}$$

then

$$x(t) := c_1 x_1(t) + c_2 x_2(t) \,:\, I \to \mathbf{R}^n \quad \text{is a solution of } x' = A(t)x + (c_1 g_1 + c_2 g_2(t))\,.$$

Proof: : This a simple exercise. ∎

This theorem has very important consequences.

Homogeneous equations. Let us consider homogeneous Cauchy problems $x' = A(t)x$, $x(t_0) = x_0$ and let denote its solutions $x(t, t_0, x_0)$ to indicate explicitly the dependence on the initial data.
(a) The solution $x(t, t_0, x_0)$ depends linearly on the initial condition x_0, i.e.,

$$x(t, t_0, c_1 x_0 + c_2 y_0) \;=\; c_1 x(t, t_0, x_0) + c_2 x(t, t_0, y_0)\,. \tag{2.2}$$

This follows by noting that, by linearity, both sides are solutions of the ODE and have the same initial conditions. The uniqueness of the solutions implies then the equality. As a consequence there exists a linear map $R(t, t_0) \,:\, \mathbf{R}^n \to \mathbf{R}^n$ such that

$$R(t, t_0)x_0 \;=\; x(t, t_0, x_0)\,. \tag{2.3}$$

It maps the initial condition x_0 at time t_0 to the position at time t. The linear map $R(t, t_0)$ is called the *resolvent* of the differential equation $x' = A(t)x$. The i-th column of $R(t, t_0)$ is a solution $x' = A(t)x$ with initial condition $x_0 = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ where 1 is in i-th position.

(b) If $x_0 = 0$, then $x(t) \equiv 0$ for all $t \in I$ (The point 0 is called a *critical point*). As a consequence if $x(t)$ is a solution and it vanishes at some point t, then it is identically 0.

(c) The set of solutions of $x' = A(t)x$ form a vector space. We call a set of solutions $x_1(t), \cdots, x_k(t)$ *linearly dependent* if there exists constants $c_1, \cdots, c_k$, with at least one $c_i \neq 0$, such that

$$c_1 x_1(t) + \cdots + c_k x_k(t) \;=\; 0\,. \tag{2.4}$$

Note that by (b), if (2.4) holds at one point t, it holds at any point t. Therefore if the initial condition $x_1(t_0), \cdots x_k(t_0)$ are linearly dependent, then the corresponding solutions are linearly dependent for any t. The k solutions are called *linearly independent* if they are not linearly dependent, i.e., $c_1 x_1(t) + \cdots + c_k x_k(t) = 0$ implies that that $c_1 = \cdots = c_k = 0$.

(d) From (c) it follows that there exist exactly n linearly independent solutions, $x_1, \cdots, x_n$. Every such set of n linearly independent solutions is called a *fundamental system* of solutions. Any solution x of $x' = A(t)x$ can be written, in a unique way, as a linear combination

$$x(t) = a_1 x_1(t) + \cdots + a_n x_n(t)\,. \tag{2.5}$$

(e) A system of n linearly independent solutions can be arranged in a matrix $\Phi(t) = (x_1(t), \cdots, x_n(t))$. In this notation the i-th column of $\Phi(t)$ is the column vector $x_i(t)$. The matrix $\Phi(t)$ is called a *fundamental matrix* or a *Wronskian* for $x' = A(t)x$. It satisfies the matrix differential equation

$$\frac{d}{dt}\Phi(t) \;=\; A(t)\Phi(t)\,. \tag{2.6}$$

(f) If $\Phi(t)$ is a fundamental matrix then the resolvent is given by

$$R(t, t_0) \;=\; \Phi(t)\Phi(t_0)^{-1}\,. \tag{2.7}$$

Indeed $x(t) = \Phi(t)\Phi(t_0)^{-1}x_0$ satisfies $x' = A(t)x$ (because of (2.6)) and $x(t_0) = x_0$.

Theorem 2.1.4 (Properties of the resolvent) *Let $A(t)$ be continuous on the interval I. Then the resolvent of $x' = A(t)x$ satisfies*

1. $\frac{\partial}{\partial t}R(t, t_0) \;=\; A(t)R(t, t_0)$.
2. $R(t_0, t_0) = I$ *(the identity matrix).*
3. $R(t, t_0) \;=\; R(t, t_1)R(t_1, t_0)$.
4. $R(t, t_0)$ *is invertible and* $R(t, t_0)^{-1} = R(t_0, t)$.

Proof: We have $\frac{\partial}{\partial t}R(t, t_0)x_0 \;=\; \frac{\partial}{\partial t}x(t, t_0, x_0) \;=\; A(t)R(t, t_0)x_0$ and $R(t_0, t_0)x_0 = x_0$ for any $x_0 \in \mathbf{R}^n$. This proves 1. and 2. Item 3 simply says that $x(t, t_0, x_0) = x(t, t_1, x(t_1, t_0, x_0))$. Item 4. follows from 2. and 3. by setting $t = t_0$.

Example 2.1.5 The harmonic oscillator $x'' + \kappa x = 0$ can be written with $x_1 = x$ and $x_2 = x'$ as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.8}$$

One can find easily two linearly independent solutions, namely

$$\begin{pmatrix} \cos(\sqrt{\kappa}t + \phi) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t + \phi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sin(\sqrt{\kappa}t + \phi) \\ \sqrt{\kappa}\cos(\sqrt{\kappa}t \end{pmatrix} \tag{2.9}$$

By definition, the resolvent is the fundamental solution $(x_1(t), x_2(t))$ with $x_1(t) = (1,0)^T$ and $x_2(t_0) = (0,1)^T$ so that we have

$$R(t,t_0) = \begin{pmatrix} \cos(\sqrt{\kappa}(t-t_0)) & \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}(t-t_0)) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}(t-t_0)) & \cos(\sqrt{\kappa}(t-t_0)) \end{pmatrix} . \qquad (2.10)$$

Note that the relation $R(t,t_0) = R(t,s)R(s,t_0)$ is simply the addition formula for sine and cosine.

Theorem 2.1.6 (Liouville) *Let $A(t)$ be continuous on the interval I and let $\Phi(t)$ be a fundamental matrix of $x' = A(t)x$. Then*

$$\det\Phi(t) = \det\Phi(t_0)\exp\left(\int_{t_0}^{t} \operatorname{trace} A(s)\, ds\right) , \qquad (2.11)$$

where $\operatorname{trace} A(t) := a_{11}(t) + \cdots + a_{nn}(t)$.

Proof: Let $\Phi(t) = (\phi_{ij}(t))_{i,j=1}^{n}$. From linear algebra we know that $\det(A)$ is a multilinear function of the rows of A. It follows that

$$\frac{d}{dt}\det\Phi(t) = \sum_{i=1}^{n}\det D_i(t) \quad \text{where} \quad D_i(t) = \begin{pmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \vdots & & \vdots \\ \phi'_{i1}(t) & \cdots & \phi'_{in}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{pmatrix} . \qquad (2.12)$$

The matrix $D_i(t)$ is obtained from $\Phi(t)$ by replacing the i-th line by its derivative. We have $\Phi'(t) = A(t)\Phi(t)$, i.e., $\phi'_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)\phi_{kj}(t)$. Using the multilinearity of the determinant we find

$$\begin{aligned} \frac{d}{dt}\det\Phi(t) &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t)\det\begin{pmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \vdots & & \vdots \\ \phi_{k1}(t) & \cdots & \phi_{kn}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{pmatrix} \longleftarrow i-\text{th line} \\ &= \left(\sum_{i=1}^{n} a_{ii}(t)\right)\det\Phi(t) . \end{aligned} \qquad (2.13)$$

This is a scalar differential which can be solved by separation of variables and gives (2.11). ∎

Remark 2.1.7 The Liouville theorem has the following useful interpretation. If $V = (v_1, \cdots, v_n)$ is a matrix whose columns are the vectors $v_1, \cdots, v_n$, then $|\det V|$ is the volume of the parallelepiped spanned by $v_1, \cdots, v_n$. Using $\det A^{-1} = 1/\det A$, Liouville Theorem is equivalent to

$$\det R(t, t_0) \; = \; \exp\left(\int_{t_0}^{t} \mathrm{trace} A(s)\, ds\right) . \tag{2.14}$$

If, at time t_0 we start with a set of initial conditions B of volume, say, 1 (e.g. a unit cube), at time t the set B is mapped to a set a parallelepiped $R(t, t_0)B$ of volume $\exp\left(\int_{t_0}^{t} \mathrm{trace} A(s)\, ds\right)$.

In particular, if $\mathrm{tr} A(t) \equiv 0$, then the flow defined by the equation $y' = A(t)y$ preserves volume. We have such a situation in Example 2.1.5, see (2.8).

Inhomogeneous equations. We consider the equation

$$x' = A(t)x + g(t) . \tag{2.15}$$

Theorem 2.1.8 *Let $\bar{x}(t)$ be a fixed solution of the inhomogeneous equation (2.15). If $x(t)$ is a solution of the homogeneous equation, then $x(t) + \bar{x}(t)$ is a solution of the homogeneous equation and all solutions of the inhomogeneous equation are obtained in this way.*

Proof: : This is an easy exercise. ∎

If we know how to solve the homogeneous problem, i.e. if we know the resolvent $R(t, t_0)$, our task is then to find just *one* solution of the inhomogeneous equation. The following theorem provides an explicit formula for such solution.

Theorem 2.1.9 (Variation of constants or Duhamel's formula) *Let $A(t)$ and $g(t)$ be continuous on the interval I and let $R(t, t_0)$ be the resolvent of the homogeneous equation $x' = A(t)x$. Then the solution of the Cauchy problem $x' = A(t)x + g(t)$ is given by*

$$x(t) \; = \; R(t, t_0)x_0 + \int_{t_0}^{t} R(t, s)g(s)\, ds . \tag{2.16}$$

Proof: The general solution of the homogeneous equation has the form $R(t, t_0)c$ with $c \in \mathbf{R}^n$. The idea is to "vary the constants" and to look for a solution of the inhomogeneous problem of the form

$$x(t) = R(t, t_0)c(t) . \tag{2.17}$$

We must then have, using 1. of Theorem 2.1.4,

$$\begin{aligned} x'(t) &= \frac{\partial}{\partial t} R'(t,t_0)c(t) + R(t,t_0)c'(t) \;=\; A(t)R(t,t_0)c(t) + R(t,t_0)c'(t) \\ &= A(t)R(t,t_0)c(t) + g(t)\,. \end{aligned} \tag{2.18}$$

Thus

$$c'(t) \;=\; R(t,t_0)^{-1}g(t) \;=\; R(t_0,t)g(t)\,, \tag{2.19}$$

and, integrating, this gives $c(t) = x_0 + \int_{t_0}^{t} R(t_0,s)g(s)\,ds$. Inserting this formula in (2.17) gives the result. ∎

It should be noted that, in general, the computation of the resolvent for $x' = A(t)x$ is not easy and can rarely be done explicitly if A depends on t.

Example 2.1.10 Forced harmonic oscillator We consider the differential equation $x'' + x = f(t)$, or equivalently the first order system $x' = y$, $y' = -x - f(t)$. The resolvent is given by (2.10). The solution of the above system with initial conditions $(x(0), y(0))^T = (x_0, y_0)^T$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \;=\; R(t)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t \begin{pmatrix} f(s)\sin(t-s) \\ f(s)\cos(t-s) \end{pmatrix} ds\,, \tag{2.20}$$

so that

$$x(t) \;=\; \cos(t)x_0 + \sin(t)y_0 + \int_0^t f(s)\sin(t-s)\,ds\,. \tag{2.21}$$

For example if $f(t) = \cos(\sqrt{\kappa}t)$ we find

$$x(t) \;=\; \cos(t)x_0 + \sin(t)y_0 + \begin{cases} \frac{\sqrt{\kappa}}{1-\kappa}(\cos(t) - \cos(\sqrt{\kappa}t)) & \kappa \neq 1 \\ \frac{1}{2}t\sin(t) & k = 1 \end{cases}\,. \tag{2.22}$$

The motion is quasi-periodic if $\sqrt{\kappa}$ is irrational, periodic if $\sqrt{\kappa}$ is rational (and $\neq 1$), and the solution grows as $t \to \infty$ if $\kappa = 1$ (resonance).

2.2 The exponential of a linear map A

In this section we let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We equip $\mathbf{K}^n$ with a norm, for example,

$$\|x\|_1 \;=\; \sum_{i=1}^{n} |x_i|\,, \quad \|x\|_2 \;=\; \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\,, \quad \|x\|_\infty \;=\; \max_{1\le i\le n} |x_i|\,, \tag{2.23}$$

then $\mathbf{K}^n$ is a Banach space. All norms being equivalent on $\mathbf{K}^n$ the choice is a matter of convenience.

A $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathbf{K}$ defines a linear map $A : \mathbf{K}^n \to \mathbf{K}^n$ and we denote by $\mathcal{L}(\mathbf{K}^n)$ the set of all linear maps from $\mathbf{K}^n$ into $\mathbf{K}^n$. The set $\mathcal{L}(\mathbf{K}^n)$ is also a vector space, of (real or complex) dimension n^2 and is a Banach space if equipped with a norm. In addition to being a vector space $\mathcal{L}(\mathbf{K}^n)$ is naturally equipped with multiplication (composition of linear maps) and it is natural and advantageous to equip $\mathcal{L}(\mathbf{K}^n)$ with a norm which is compatible with matrix multiplication.

Definition 2.2.1 For $A \in \mathcal{L}(\mathbf{K}^n)$ we define

$$\|A\| \;=\; \sup_{\|x\|\le 1} \|Ax\| \;=\; \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}\,. \tag{2.24}$$

The number $\|A\|$ is called the *operator norm* of A.

This definition means that $\|A\|$ is the smallest real number R such that

$$\|Ax\| \;\le\; R\,\|x\|\,, \qquad \text{for all } x \in \mathbf{K}^n\,, \tag{2.25}$$

and we have the bound

$$\|Ax\| \;\le\; \|A\|\|x\|\,. \tag{2.26}$$

The properties **N1** and **N2** are easily verified. For the triangle inequality, we have for $A, B \in \mathcal{L}(\mathbf{K}^n)$

$$\|(A+B)x\| \le \|Ax\| + \|Bx\| \;\le\; (\|A\| + \|B\|)\,\|x\|\,. \tag{2.27}$$

Dividing by $\|x\|$ and taking the supremum over all $x \neq 0$ one obtains the triangle inequality $\|A+B\| \le \|A\| + \|B\|$.

Simple but important properties of $\|A\|$ are summarized in

Lemma 2.2.2 *Let $I \in \mathcal{L}(\mathbf{K}^n)$ be the identity map ($Ix = x$) and let $A, B \in \mathcal{L}(\mathbf{K}^n)$. Then we have*

1. $\|I\| = 1$.
2. $\|AB\| \le \|A\|\,\|B\|$. .
3. $\|A^n\| \le \|A\|^n$.

Proof: 1. is immediate, 3. is a consequence of 2. To estimate $\|AB\|$, we apply twice (2.25)

$$\|(AB)x\| \;\le\; \|A\|\,\|Bx\| \;\le\; \|A\|\|B\|\|x\|\,. \tag{2.28}$$

To conclude we divide by $\|x\|$ and take the supremum over $x \neq 0$. ∎

Example 2.2.3 Let us denote by $\|A\|_p$, $p = 0, 1, \infty$ the operator norm of A acting on $\mathbf{K}^n$ with the norm $\|x\|_p$, $p = 0, 1, \infty$ (see (2.23)). Then we have the formulas

$$\begin{aligned} \|A\|_1 &= \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|\,, \\ \|A\|_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|\,, \\ \|A\|_2 &= \sqrt{\text{biggest eigenvalue of } A^*A}\,. \end{aligned} \tag{2.29}$$

Proof: For $\|x\|_1$ we have

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \le \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|\,|x_j| = \sum_{j=1}^{n} |x_j| \left(\sum_{i=1}^{m} |a_{ij}| \right) \le \max_{1 \le j \le n} \left(\sum_{i=1}^{m} |a_{ij}| \right) \|x\|_1\,, \tag{2.30}$$

and therefore $\|A\|_1 \le \max_j (\sum_{i=1}^m |a_{ij}|)$. To prove the equality, choose j_0 such that $\sum_{i=1}^m |a_{ij_0}| = \max_j (\sum_{i=1}^m |a_{ij}|)$ and then set $x = (0, \cdots, 1, \cdots, 0)^T$ where the 1 is in position j_0. Then for such x we have equality in (2.30). This shows that $\|A\|_1$ cannot be smaller than $\max_j (\sum_{i=1}^m |a_{ij}|)$. The formula for $\|A\|_\infty$ is proved similarly.

For the norm $\| \cdot \|_2$ we have $\|x\|_2^2 = \langle x, x \rangle$ where $\langle x, y \rangle = \sum_{i=1}^n \bar{x}_i y_i$ is the usual scalar product. Note that the matrix A^*A is symmetric and positive semi-definite ($\langle x, A^*Ax \rangle = \|Ax\|_2 \ge 0$). From linear algebra we know that A^*A can be diagonalized and there exists an unitary matrix U ($U^*U = 1$) such that $U^*A^*AU = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, where $\lambda_i \ge 0$. With $x = Uy$ ($\|x\|_2 = \|y\|_2$) we obtain

$$\|Ax\|_2^2 \;=\; \langle x, A^*Ax \rangle \;=\; \langle y, U^*A^*AUy \rangle \;=\; \sum_{i=1}^{n} \lambda_i |y_i|^2 \;\le\; \lambda_{\max} \|y\|_2^2 \;=\; \|x\|_2^2\,. \tag{2.31}$$

This implies that $\|A\|_2 \le \sqrt{\lambda_{\max}}$. To show equality choose x to be an eigenvector for $\lambda_{\max}$. ∎

In order to solve linear ODE's we will need to construct the exponential of a $n \times n$ matrix A which we will denote by e^A. We will define it using the series representation of the exponential function. If $\{C_k\}$ is a sequence with $C_k \in \mathcal{L}(\mathbf{K}^n)$ we define infinite series as usual: $C = \sum_{k=0}^\infty C_k$ if and only if the partial sums converge. The convergence of $C = \sum_{k=0}^\infty C_k$ is equivalent to the convergence of the n^2 series of the coefficients $\sum_k c_{ij}^{(k)}$. We say that the series converges absolutely if the real series $\sum \|C_k\|$ converges. For any norm, there exist positive constants a and A such that $a \sum_{ij} |c_{ij}^{(k)}| \le \|C_k\| \le A \sum_{ij} |c_{ij}^{(k)}|$ and therefore absolute convergence of the series is equivalent to the absolute convergence of the n^2 series $\sum_k c_{ij}^{(k)}$.

Proposition 2.2.4 *Let $A \in \mathcal{L}(\mathbf{K}^n)$. Then*

1. *For any $T > 0$, the series*
$$e^{tA} := \sum_{j=0}^{\infty} \frac{t^j A^j}{j!}\,, \tag{2.32}$$
converges absolutely and uniformly on $[-T, T]$, e^{tA} is a continuous function of t and we have
$$\left\| e^{tA} \right\| \le e^{t\|A\|}\,. \tag{2.33}$$

2. *The map $t \to e^{tA}$ is everywhere differentiable and*
$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A\,. \tag{2.34}$$

Proof: Item 1. For $t \in [-T, T]$ we have
$$\left\| \frac{t^j A^j}{j!} \right\| \le \frac{|t|^j \|A\|^j}{j!} \le \frac{T^j \|A\|^j}{j!}\,. \tag{2.35}$$

Let us denote by $S_n(t)$ the partial sum $\sum_{j=0}^{n} \frac{t^j A^j}{j!}$. Then, for $m > n$, we have
$$\|S_n(t) - S_m(t)\| \le \sum_{j=n+1}^{m} \frac{T^j \|A\|^j}{j!} \le \sum_{j=n+1}^{\infty} \frac{T^j \|A\|^j}{j!}\,. \tag{2.36}$$

This implies that $S_n(t)$ is a Cauchy sequence in $\mathcal{L}(\mathbf{K}^n)$, uniformly in $t \in [-T, T]$, since the right side (2.36) is the remainder term for the series $e^{T\|A\|}$. The function $S_n(t)$ are continuous function, they converge uniformly on $[-T, T]$ and $\mathcal{L}(\mathbf{K}^n)$ is a Banach space so that the limit e^{tA}, exists and is continuous. The bound (2.33) follows immediately from (2.35).

Item 2. The partial sum $S_n(t)$ are differentiable function of t with
$$S_n'(t) = A S_{n-1}(t) = S_{n-1}(t) A\,. \tag{2.37}$$

The same argument as in 1. shows that $S_n'(t)$ converges uniformly on $[-T, T]$. Since both $S_n(t)$ and $S_n'(t)$ converge uniformly we can exchange limit and differentiation. If we take the limit $n \to \infty$ in (2.37) we obtain (2.34). ∎

We summarize some properties of the exponential in

Proposition 2.2.5 *Let $A, B, C \in \mathcal{L}(\mathbf{K}^n)$. Then*

1. *If $AB = BA$ then $e^{A+B} = e^A e^B$.*

2. *If C is invertible then $e^{C^{-1}AC} = C^{-1} e^A C$.*

3. *If* $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ *then* $e^A = \mathrm{diag}(e_1^\lambda, \cdots, e_n^\lambda)$.

Proof: If $AB = BA$ then, using the binomial theorem, we obtain

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \tag{2.38}$$

and therefore

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{A^k B^{n-k}}{k!(n-k)!} = \sum_{p=0}^{\infty} \frac{A^p}{p!} \sum_{q=0}^{\infty} \frac{B^q}{q!} = e^A e^B\,, \tag{2.39}$$

and this proves 1.

It is easy to see that $C^{-1}A^kC = (C^{-1}AC)^k$. Dividing by $k!$, summing and taking the limit proves 2. If $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then $A^k = \mathrm{diag}(\lambda_1^k, \cdots, \lambda_n^k)$ and this proves 3. ∎

As a consequence we obtain

Corollary 2.2.6 *Let* $A \in \mathcal{L}(\mathbf{K}^n)$. *Then*

1. $(e^{tA})^{-1} = e^{-tA}$.

2. $e^{(t+s)A} = e^{tA}e^{sA}$.

3. $e^{\lambda I + A} = e^{\lambda}e^{A}$.

Example 2.2.7 Let us compute the exponential of some simple matrices.

1. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $J^2 = -I$ and thus by induction $J^{2n} = (-1)^n I$ and $J^{2n+1} = (-1)^n J$. We obtain

$$e^{tJ} = \sum_{n\geq 0} \frac{(-1)^n t^{2n}}{2n!} I + \sum_{n\geq 0} \frac{(-1)^n t^{2n+1}}{(2n+1)!} A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \tag{2.40}$$

2. Let $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ then using 1. we have $e^{tA} = e^{tbJ} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$.

3. Let $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ then we have $A = aI + bJ$ and I and J commute. Thus

$$e^{tB} = e^{taI}e^{tbJ} = \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}. \tag{2.41}$$

4. Let $C = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$ then $C^2 = 0$ and thus $e^{tC} = \begin{pmatrix} 1 & \epsilon t \\ 0 & 1 \end{pmatrix}$

5. Let D be the $n \times n$ matrix with entries ϵ on the off-diagonal and) otherwise

$$D = \begin{pmatrix} 0 & \epsilon & & & \\ & 0 & \epsilon & & \\ & & \ddots & & \\ & & & 0 & \epsilon \\ & & & & 0 \end{pmatrix}, \tag{2.42}$$

We have

$$D^2 = \begin{pmatrix} 0 & 0 & \epsilon^2 & & & \\ & 0 & 0 & \epsilon^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & \epsilon^2 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} \cdots D^{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & & \epsilon^{n-1} \\ & 0 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}. \tag{2.43}$$

and $D^l = 0$ for $l \geq n$. Then we have

$$\begin{aligned} e^{tD} &= 1 + tD + \frac{t^2 D^2}{2!} + \cdots + \frac{t^{n-1} D^{n-1}}{(n-1)!} \\ &= \begin{pmatrix} 1 & \epsilon t & \frac{\epsilon^2 t^2}{2} & \cdots & & \frac{\epsilon^{n-1} t^{n-1}}{(n-1)!} \\ & 1 & \epsilon t & \cdots & & \frac{\epsilon^{n-2} t^{n-2}}{(n-2)!} \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & \epsilon t & \frac{\epsilon^2 t^2}{2} \\ & & & & 1 & \epsilon t \\ & & & & & 1 \end{pmatrix} \end{aligned} \tag{2.44}$$

6. Let

$$E = \begin{pmatrix} \lambda & \epsilon & & & \\ & \lambda & \epsilon & & \\ & & \ddots & & \\ & & & \lambda & \epsilon \\ & & & & \lambda \end{pmatrix},$$

then

$$e^{tE} = e^{t(\lambda I + D)} = e^{\lambda t} e^{tD} \tag{2.45}$$

$$= \begin{pmatrix} e^{\lambda t} & \epsilon t e^{\lambda t} & \frac{\epsilon^2 t^2}{2} e^{\lambda t} & \cdots & & \frac{\epsilon^{n-1} t^{n-1}}{(n-1)!} e^{\lambda t} \\ & e^{\lambda t} & \epsilon t e^{\lambda t} & \cdots & & \frac{\epsilon^{n-2} t^{n-2}}{(n-2)!} e^{\lambda t} \\ & & \ddots & \ddots & & \vdots \\ & & & e^{\lambda t} & \epsilon t e^{\lambda t} & \frac{\epsilon^2 t^2}{2} e^{\lambda t} \\ & & & & e^{\lambda t} & \epsilon t e^{\lambda t} \\ & & & & & e^{\lambda t} \end{pmatrix} \tag{2.46}$$

2.3 Linear systems with constant coefficients

From Proposition 2.2.4 one obtains immediately

Theorem 2.3.1 *The resolvent of the linear equation with constant coefficients $x' = Ax$ is given by*

$$R(t, t_0) = e^{(t-t_0)A}. \tag{2.47}$$

Proof: From proposition 2.2.4 we have

$$\frac{d}{dt} e^{(t-t_0)A} = A e^{(t-t_0)A}. \tag{2.48}$$

Thus the j-th column of $e^{(t-t_0)A}$ is the solution of the Cauchy problem $x' = Ax$, $x(t_0) = (0, \cdots, 0, 1, 0, \cdots)^T$ where the 1 is in j-th position. ∎

Solving $x' = Ax$ is thus reduced to the problem of computing the exponential of a matrix A, see Example 2.2.7 for some simple examples. We will present here a general technique to compute such an exponential.

In the scalar case the ODE $x' = \lambda x$ has the general solution $X(t) = Ce^{\lambda t}$. With this intuition in mind let us try to find solutions of the form $x(t) = e^{\lambda t} v$ where v is a nonzero vector. Inserting into the equation we deduce that $e^{\lambda t} v$ is a solution if and only if

$$Av = \lambda v, \tag{2.49}$$

i.e., λ is an eigenvalue of $A$and v is an eigenvector for the eigenvalue λ.

If A is real and λ is a complex eigenvalue with eigenvector $w = u + iv$ then we have $A\bar{w} = \bar{\lambda}\bar{w}$, i.e. the eigenvalues and eigenvectors occur in complex conjugate pairs.

Proposition 2.3.2 *Let A be a real $n \times n$ matrix and consider the differential equation $x' = Ax$.*

1. *The function $t \mapsto e^{\lambda t} v$ is a real solution if and only if $\lambda \in \mathbf{R}$, $v \in \mathbf{R}^n$, and $Av = \lambda v$.*

2. *If $w \neq 0$ is an eigenvector for A with eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$ then the imaginary part of $w = u + iv$ is not zero. In this case there are two real solutions*

$$t \mapsto e^{\alpha t}\left[(\cos\beta t)u - (\sin\beta t)v\right] , \tag{2.50}$$

$$t \mapsto e^{\alpha t}\left[(\sin\beta t)u + (\cos\beta t)v\right] . \tag{2.51}$$

Proof: If $Av = \lambda v$ then $e^{\lambda t}v$ is a solution. If $\lambda = \alpha + i\beta$, then since A is real an eigenvector $w = u + iv$ has nonzero imaginary part. The real and imaginary parts of the corresponding solution

$$\begin{aligned} e^{\lambda t}(u+iv) &= e^{(\alpha+i\beta)t}(u+iv)\,, \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(u+iv)\,, \\ &= e^{\alpha t}\left[(\cos\beta t)u - (\sin\beta t)v\right] + ie^{\alpha t}\left[(\sin\beta t)u + (\cos\beta t)v\right] . \end{aligned} \tag{2.52}$$

are real solutions. In order to show that these real solutions are linearly independent, let us suppose that some linear combinations of them vanishes identically. Evaluating at $t = 0$ and $t = \pi/2\beta$ yields

$$c_1 u + c_2 v = 0 \qquad c_2 u - c_1 v \;=\; 0\,, \tag{2.53}$$

This implies that $(c_1^2 + c_2^2)w = 0$ and thus $c_1 = c_2 = 0$. This proves item 2. The proof of item 1. is easy. ∎

The problem now is reduced to the question whether we can find n linearly independent eigenvectors of A. As we know from linear algebra this is not always possible, for example the matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \tag{2.54}$$

has 1 as its only eigenavalue and $(1,0,0)^T$ as its only eigenvector.

Definition 2.3.3 Let Λ be an eigenvalue of A then we define

(i) The eigenvalue λ of A has an *algebraic multiplicity* equal to l if λ is a zero of order l of the charateristic polynomial $\det(A - \lambda I)$.

(ii) The eigenvalue λ of A has a *geometric multiplicity* equal to k if k is the dimension of the subspace spanned by the eigenvectors of A for the eigenvalue λ, i.e. $k = \dim(\ker(A - \lambda I))$.

The algebraic multiplicity of λ for the matrix A given by (2.54) is 3 but its algebraic multiplicity is 1.

Definition 2.3.4 The matrix A is called *semi-simple* or *diagonalizable* if for each eigenvalue λ algebraic and geometric multiplicity coincide.

In this case it is, in principle, easy to compute e^{At}, we have

Proposition 2.3.5 *Let A be a semi-simple $n \times n$ matrix (real or complex) with eigenvalues $\lambda_1, \cdots, \lambda_n$ repeated according to their algebraic multplicity then there exists a basis $v_1, \cdots v_n$ of $\mathbf{C}^n$ where v_i is an eigenvectors of A for the eigenvalue Λ_i. Let*

$$P = (v_1, \cdots, v_n) \tag{2.55}$$

be the matrix whose i^{th} column is given by the vector v_i. Then

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} . \tag{2.56}$$

Proof: We have

$$AP = (Av_1, \cdots, Av_n) = (\lambda_1 v_1, \cdots, \lambda_n v_n) = (v_1, \cdots, v_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} , \tag{2.57}$$

i.e., $D \equiv P^{-1}AP$ is a diagonal matrix whose entries are the eigenvalues of A. Then the resolvent e^{At} for $x' = Ax$ is given by

$$e^{At} = PP^{-1}e^{At}PP^{-1} = Pe^{P^{-1}APt}P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} . \tag{2.58}$$

∎

Remark 2.3.6 The change of variable $y = P^{-1}x$ transform the system $x' = Ax$ into a system of decoupled equations. Indeed we have $y' = P^{-1}x' = P^{-1}Ax == Dy$ where D is diagonal. Thus we have n equations $y_j' = \lambda_j y_j$ whose solutions $e^{\lambda_j t}v_j$ form a fundamental matrix for $y' = Dy$.

Example 2.3.7

$$\begin{array}{rcrcrcr} x_1' &=& x_1 &-& 2x_2 & & \\ x_2' &=& 2x_1 & & &-& x_3 \\ x_3' &=& 4x_1 &-& 2x_2 &-& x_3 \end{array} , \qquad A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix} . \tag{2.59}$$

The eigenvalues are the root of $\det(A-\lambda I) = 2-\lambda-\lambda^3 = (1-\lambda)(\lambda^2+\lambda+2)$. The eigenvalues are $\lambda_{1,2} = -1/2 \pm i\sqrt{7}/2$, and $\lambda_3 = 1$. The eigenvectors are computed to be $v_{1,2} = (3/2 \pm i\sqrt{7}/2, 2, 4)^T$ and $v_3 = (1,0,2)^T$ and $B = (v_1, v_2, v_3)$. Three real linearly independent solutions are given by

$$e^t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{2.60}$$

$$e^{-t/2}\left[\cos(\sqrt{7}t/2)\begin{pmatrix} 3/2 \\ 2 \\ 4 \end{pmatrix} - \sin(\sqrt{7}t/2)\begin{pmatrix} \sqrt{7}/2 \\ 0 \\ 0 \end{pmatrix}\right] \tag{2.61}$$

$$e^{-t/2}\left[\sin(\sqrt{7}t/2)\begin{pmatrix} 3/2 \\ 2 \\ 4 \end{pmatrix} + \cos(\sqrt{7}t/2)\begin{pmatrix} \sqrt{7}/2 \\ 0 \\ 0 \end{pmatrix}\right] \tag{2.62}$$

Suppose that A is not semi-simple, i.e., if, for at least one eigenvalue, the geometric multiplicity is smaller than the algebraic multiplicity. One way to solve the system is to transform A into a simpler form, for example in a triangular form or in Jordan normal form, i.e., one finds an invertible T such that $T^{-1}AT$ has such form (see the exercises for a proof than any matrix can be transformed into triangular form). With the transformation $x = Ty$ and $x' = Ty'$ the ODE $x' = Ax$ becomes $y' = Sy$. For example if S has a triangular form we have the system

$$\begin{array}{lcccccccc} y_1' & = & s_{11}y_1 & + & s_{12}y_2 & + & \cdots & + & s_{1n}y_n \\ y_2' & = & & & s_{22}y_2 & + & \cdots & + & s_{2n}y_n \\ \vdots & & & & & & & & \vdots \\ y_n' & = & & & & & & & s_{nn}y_n \end{array} \tag{2.63}$$

One can then solve the system iteratively: one solves first the equation for y_n, then the one for y_{n-1}, and so on up to the equation for y_1 (see the example below). Finally one obtains $x = Ty$.

Example 2.3.8 Consider the system of equations

$$\begin{array}{lcccccc} x_1' & = & -3x_1 & + & 2x_2 & + & 5x_3 \\ x_2' & = & & + & x_2 & - & x_3 \\ x_3' & = & & & & & 2x_3 \end{array}, \qquad A = \begin{pmatrix} -3 & 2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.64}$$

with initial conditions $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30})$. The third equations has solution $x_3(t) = e^{2t}x_{30}$. Inserting into the second equations gives the inhomogeneous equation $x_2' = x_2 - e^{2t}x_{30}$ and is solved using Duhamel's formula

$$x_2(t) = e^t x_{20} - \int_0^t e^{t-s}e^{2s}x_{30} = e^t x_{20} + (e^t - e^{2t})x_{30}\,. \tag{2.65}$$

Inserting the solutions $x_2(t)$ and $x_3(t)$ into the first equations gives the equation $x_1' = -3x_1 + 2e^t x_{20} + (2e^t + 3e^{2t})x_{30}$. Again with Duhamel's formula one finds

$$x_1(t) \;=\; e^{-3t}x_{10} + 2(e^t - e^{-3t})x_{20} + (2e^t + 3e^{2t} - 5e^{-3t})x_{30}\,. \tag{2.66}$$

The resolvent is then

$$R(t,t_0) \;=\; \begin{pmatrix} e^{-3(t-t_0)} & 2e^{(t-t_0)} - 2e^{-3(t-t_0)} & 2e^{(t-t_0)} + 3e^{2(t-t_0)} - 5e^{-3(t-t_0)} \\ 0 & e^{(t-t_0)} & e^{(t-t_0)} - e^{2(t-t_0)} \\ 0 & 0 & e^{2(t-t_0)} \end{pmatrix}. \tag{2.67}$$

The resolvent can be computed easily if S is in Jordan normal form. Let us consider first the complex Jordan normal form. Then S is block diagonal

$$S \;=\; \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} \tag{2.68}$$

where each Jordan block J_i has the form

$$J_i \;=\; \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix} \tag{2.69}$$

Since S is block diagonal we have

$$e^{tS} \;=\; \begin{pmatrix} e^{tJ_1} & & \\ & \ddots & \\ & & e^{tJ_k} \end{pmatrix}, \tag{2.70}$$

so that it is enough to compute e^{tJ} where J is a Jordan block. This has been computed in Example 2.2.7 5, and 6.

Example 2.3.9 The system of equations

$$\begin{array}{lcl} x_1' &=& -2x_1 + x_2 \\ x_2' &=& + \; -2x_2 \\ x_3' &=& 2x_3 \end{array}, \tag{2.71}$$

is already in Jordan normal form and its resolvent is

$$e^{tS} \;=\; \begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}. \tag{2.72}$$

There is also a real Jordan normal form if A is a real matrix. If $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$ are a pair complex eigenvalues then S has the form (2.68) where the block corresponding to the pair λ, $\bar{\lambda}$ is given by

$$J = \begin{pmatrix} R & I & & & \\ & R & I & & \\ & & \ddots & \ddots & \\ & & & R & I \\ & & & & R \end{pmatrix} = \begin{pmatrix} R & 0 & & & \\ & R & 0 & & \\ & & \ddots & \ddots & \\ & & & R & 0 \\ & & & & R \end{pmatrix} + \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ & & & & 0 \end{pmatrix}$$
$$\equiv \quad T \quad + \quad M$$

where I is a 2×2 identity matrix and R has the form

$$R = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} . \tag{2.73}$$

The exponential of R is given by $e^{Rt} = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$. Noting that T commute with M we have that $e^{Jt} = e^{Tt}e^{Mt}$ and this can be computed easily.

We will not discuss in detail here the algorithm used to put the matrix in Jordan normal form, since this is not necessary to compute the resolvent e^{At}. We will use a slightly simpler algorithm to compute e^{At}. It is based on a fundamental result of linear algebra which we quote here without proof.

Definition 2.3.10 *Let λ be an eigenvalue of A. The* generalized eigenspace *of λ consists of the subspace*

$$E_\lambda = \{v \,;\, (A - \lambda I)^k v = 0 \,, \text{ for some } k \geq 1\} \tag{2.74}$$

The elements of the generalized eigenspace are called generalized eigenvectors

Note that if A is semi-simple the generalized eigenspace are obtained by taking only $k = 1$ and thus consist only of eigenvectors.

We will need the following simple result

Lemma 2.3.11 *The generalized eigenspace E_λ is invariant under A.*

Proof: If $v \in E_\lambda$ then $(A - \lambda I)^k v = 0$. Then

$$(A - \lambda I)^k Av = (A - \lambda I)^k Av - \lambda(A - \lambda I)^k v = (A - \lambda)(A - \lambda I)^k v = 0 \tag{2.75}$$

and thus $Tv \in E_\lambda$.

We have the fundamental result

Theorem 2.3.12 *Let A be a $n \times n$ matrix. Then there exists a basis of $\mathbf{C}^n$ which consists of generalized eigenvectors, i.e.,*

$$\mathbf{C}^n = \bigoplus_{\lambda \text{ eigenvalues}} E_\lambda \tag{2.76}$$

Using this we will show that any matrix A can be decomposed into a semi-simple part and a nilpotent part. An example of nilpotent matrix is given in Example 2.2.7.

Definition 2.3.13 A matrix N is said to be *nilpotent with nilpotency k* if $N^k = 0$ but $N^{k-1} \neq 0$

Proposition 2.3.14 *Let A be a $n \times n$ matrix, then there exists a decomposition*

$$A = S + N \tag{2.77}$$

where A is semi-simple, N is nilpotent and commute with A and with nilpotency no larger than the maximum of the algebraic multiplicities of the eigenvalues.

Proof: Let $v_1, \cdots, v_n$ be a basis consisting of generalized eigenvectors and set $P = (v_1, \cdots, v_n)$ be the matrix whose i^{th} column is v_i. Let $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$ be the diagonal matrix where $\lambda_i = \lambda$ if $v_i \in E_\lambda$. Then we define

$$S \equiv P\Lambda P^{-1}, \quad N \equiv A - S. \tag{2.78}$$

This provides a decomposition $S = A + N$.

By construction S is semi-simple and has the same eigenvalues as A with the same algebraic multiplicities.

Note that $SN - NS = S(A - S) - (A - S)S = SA - AS$ and so it is enough to show that S commutes with A. Let $v \in E_\Lambda$ then $Sv = \lambda v$. Moreover $AV \in E_\lambda$ by Lemma 2.3.11 and thus Av is an eigenvector for S. So we have

$$(SA - AS)v \;=\; SAv - A\lambda v \;=\; (S - \lambda I)Av \;=\; 0\,. \tag{2.79}$$

By Theorem 2.3.12 any $v \in \mathbf{C}^n$ can be written as a sum of generalized eigenvectors and thus

$$(SA - AS)v = 0\,. \tag{2.80}$$

for any $v \in \mathbf{C}^n$ so $SA - AS = 0$ and so $SN - NS = 0$.

Finally we show that N is nilpotent. Let choose m to be larger than the largest algebraic multiplicity of the eigenvalues of A. If $v \in E_\lambda$ we have $Sv = \lambda v$ and thus using that S commute with A we obtain

$$N^m v \;=\; (A-S)^m v \;=\; (A-S)^{m-1}(A-\lambda I)v \;=\; (A-\lambda I)(A-S)^{m-1} \;=\; (A-\lambda I)^m v \;=\; 0\,. \tag{2.81}$$

Since any v can be written as a sum of generalized eigenvectors we obtain $N^m v = 0$ for any $v \in \mathbf{C}^n$ and so $N^m = 0$.

This concludes the proof of Proposition 2.3.14. ∎

Note that Proposition 2.3.14 provides a algorithm to compute e^{tA}.

Example 2.3.15 Let $x' = Ax$ with

$$A = \begin{pmatrix} -2 & -1 & -2 \\ -2 & -2 & -2 \\ 2 & 1 & 2 \end{pmatrix} . \tag{2.82}$$

We have $\det(A - \lambda I)\lambda^2(\lambda + 2)$ and so $\lambda = 0$ has algebraic multiplicity 2 and $\lambda = -2$ has algebraic multiplcity 1. The vector $(1, 2, -1)^T$ is an eigenvector for -2. We have

$$(A - 0I)^2 \;=\; A^2 \;=\; \begin{pmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ -2 & -2 & -2 \end{pmatrix} \tag{2.83}$$

and so we can choose $(1, 0, -1)^T$ and $(0, 1, -1)^T$ has generalized eigenvectors. We obtain

$$P \;=\; \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} , \qquad P^{-1} \;=\; \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & 0 & -2 \end{pmatrix} \tag{2.84}$$

and

$$\Lambda \;=\; \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad S \;=\; P\Lambda P^{-1} \;=\; \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \tag{2.85}$$

$$N \;=\; A - S \;=\; \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} . \tag{2.86}$$

Finally we compute the exponential by

$$e^{tA} \;=\; Pe^{t\Lambda}P^{-1}(I + tN) \;=\; \frac{1}{2} \begin{pmatrix} e^{-2t} + 1 - 2t & e^{-2t} - 1 & e^{-2t} - 1 - 2t \\ 2e^{-2t} - 2 & 2e^{-2t} & 2e^{-2t} - 2 \\ -e^{-2t} + 1 + 2t & -e^{-2t} + 1 & -e^{-2t} + 3 + 2t \end{pmatrix} \tag{2.87}$$

A simple but important consequence of this decomposition is the following

Proposition 2.3.16 *If A is a real $n \times n$ matrix, then e^{tA} is a matrix whose components are sums of terms of the form $p(t)e^{\alpha t} \sin \beta t$ and $p(t)e^{\alpha t} \cos \beta t$ where α are real numbers such that $\lambda = \alpha + i\beta$ is an eigenvalue of A and $p(t)$ is a polynomial of degree at most $n - 1$.*

2.4 Stability of linear systems

For the ODE, $x' = f(t, x)$ we say that x_0 is a *critical point* if $f(t, x_0) = 0$ for all t. This implies that the constant solution $x(t) = x_0$ is a solution of the Cauchy problem with $x(t_0) = x_0$. Critical points are also called *equilibrium points*.

For a linear system with constant coefficients, i.e., $f(x) = Ax$, $x_0 = 0$ is always critical point and it is the only critical point if $\det(A) \neq 0$. If $\det(A) = 0$, then 0 is an eigenvalue and any point in the eigenspace of the eigenvalue 0 is a critical point.

We define next the concept of *stability* of a solution

Definition 2.4.1 *Let $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and locally Lipschitz. Let $x(t, t_0, x)$ be the solution of the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$ which we assume to exist for all times $t > t_0$.*

1. *The solution $x(t, t_0, x_0)$ is* stable *(in the sense of Liapunov) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all ξ with $\|\xi\| \leq \delta$ we have*
$$\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| \leq \epsilon \quad \text{for all } t \geq t_0 \,. \tag{2.88}$$

2. *The solution $x(t, t_0, x_0)$ is* asymptotically stable *if it is stable and there exists $\delta > 0$ such that hat for all ξ with $\|\xi\| \leq \delta$ we have*
$$\lim_{t\to\infty} \|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| = 0 \,. \tag{2.89}$$

3. *The solution $x(t, t_0, x_0)$ is* unstable *if it is not stable.*

If a is a critical point, we will say the critical point a is stable or unstable if the solution $x(t) \equiv a$ is stable or unstable.

Example 2.4.2 The solution $x(t) = 0$ of $x' = \lambda x$ is asymptotically stable of $\lambda < 0$, stable if $\lambda = 0$, unstable if $\lambda > 0$.

Example 2.4.3 The solutions of the equation $x'' + x = 0$ are stable (but not asymptotically stable). The general solution is $(a\cos(t) + b\sin(t), -a\sin(t) + b\cos(t))$ is a periodic solution of period 2π on the circle of radius $a^2 + b^2$. Two solutions starting at nearby points (x_1, y_1) and (x_0, y_0) will remain close forever.

Example 2.4.4 The solution $x(t, 0, x_0) = \frac{x_0}{1-x_0 t}$ of $x' = x^2$ is asymptotically stable for $x_0 < 0$ but unstable for $x_0 \geq 0$.

Example 2.4.5 For the ODE $x' = x^2 - 1$, there are two critical points 0 and 1. The solution $x(t) = 0$ is unstable and the solution $x(t) = 1$ is stable.

For a linear homogeneous equation $x' = A(t)x$ we have $x(t, t_0, x_0+\xi)-x(t, t_0, x_0) = R(t, t_0)\xi = x(t, t_0, \xi) - x(t, t_0, 0)$ and so it suffices to study the stability of the critical point 0. For a linear inhomogeneous equation $x' = A(t)x + f(t)$, the difference $x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)$ is again equal to $R(t, t_0)\xi$ where $R(t, t_0)$ is the resolvent of the homogeneous equation $x' = A(t)x$ and thus the stability properties of a solution $x(t)$ of the inhomogeneous problem are the same as the stability of the trivial solution of the homogeneous problem. Therefore, in the case of linear differential equations, all the solutions have the same stability properties and one can talk about the stability of the differential equation.

As we have seen in Section 2.3, the solutions of linear systems with constant coefficients are determined by the eigenvalues, and the generalized eigenvectors of the matrix A. We define the *stable, unstable and center subspaces*, denoted respectively by E^s, E^u, and E^c and defined by

$$E^s = \bigoplus_{\lambda : \mathrm{Re}\lambda<0} E_i\,, \tag{2.90}$$

$$E^u = \bigoplus_{\lambda : \mathrm{Re}\lambda>0} E_i\,, \tag{2.91}$$

$$E^c = \bigoplus_{\lambda : \mathrm{Re}\lambda=0} E_i\,. \tag{2.92}$$

$$\tag{2.93}$$

By Lemma 2.3.11 amd Theorem 2.3.12 the generalized eigenspaces span the whole space and are invariant under A and thus also under e^{tA}. So we have

$$\mathbf{R}^n = E^s \oplus E^u \oplus E^c\,, \tag{2.94}$$

and

$$e^{At}E^{\#} = E^{\#}\,, \qquad \text{for all } t \in \mathbf{R}\,, \quad \# = s, u, c\,. \tag{2.95}$$

From the proposition 2.3.16 we obtain

Theorem 2.4.6 *Let $x' = Ax$ be a linear system with constant coefficients $x' = Ax$, and let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of A.*
(a) *The critical point 0 is asymptotically stable if and only if all the eigenvalues of A have a negative real part: $\mathrm{Re}\,\lambda_i < 0$ for $i = 1, \cdots, k$, i.e., if $E^s = \mathbf{R}^n$.*
(b) *The critical point 0 is stable if and only*

1. *All the eigenvalues have a nonpositive real part $\mathrm{Re}\,\lambda_i \le 0$ for $i = 1, \cdots, k$, i.e., $E^s \oplus E^c = \mathbf{R}^n$.*

2. *If $\mathrm{Re}\lambda_i = 0$ the Jordan blocks have dimension 1.*

Proof: If $E^s = \mathbf{R}^n$ then any solutions has components which are linear combinations of terms of the form $t^k e^{\alpha t}\sin(bt)$ and $t^k e^{\alpha t}\cos(bt)$ with $a < 0$. In particular since $\lim_{t\to\infty} t^k e^{\alpha t}\sin(bt) = 0$ we see that every solution goes to 0. This implies that $\lim_{t\to\infty}\|e^{tA}\| = 0$ and so 0 is asymptotically stable.

If If $E^s \otimes E^c = \mathbf{R}^n$ then any solutions $t^k e^{\alpha t}\sin(bt)$ and $t^k e^{\alpha t}\cos(bt)$ with $a \le 0$. If E^c is non trivial there will be some terms with $a = 0$ and those terms will remain bounded only if there are no polynomial factors t^k,in those terms, i.e., only if the restriction of A to E^c is semisimple. In this case we have then $\|e^{At}\| \le K$ and thus 0 is stable. If the restriction of A to E^c has a non-trivial nilpotent there will be terms which diverge as $t \to \infty$ and 0 is not stable.

If some eigenvalue λ has a positive real part, then there exists solutions $x(t)$ with $\|x(t)\| \to \infty$ as $t \to \infty$. In this case 0 is unstable. ∎

The qualitative behavior of solutions of linear systems with constant coefficients is as follows

- If $x \in E^s$ then $x(t) = e^{At}x$ satisfies $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to-\infty}\|x(t)\| = \infty$.
- If $x \in E^u$ then $x(t) = e^{At}x$ satisfies $\lim_{t\to\infty}\|x(t)\| = \infty$ and $\lim_{t\to-\infty} x(t) = 0$.
- If $x \in E^c$ then $x(t)$ either stays bounded for all $t \in \mathbf{R}$ or $\lim_{t\to\pm\infty}\|x(t)\| = \infty$.

We illustrate the behavior of solutions for linear 2 dimensional systems with constant coefficients in the following figures. In figure 2.1 we show the 2 stable linear systems ($E^s = \mathbf{R}^2$) with distinct eigenvalues and Jordan normal forms

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \alpha < 0\,, \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1 < 0, \lambda_2 < 0\,.$$

and in figure 2.2 the 2 stable linear systems ($E^s = \mathbf{R}^2$) with one eigenvalue and Jordan normal forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda < 0\,, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda < 0\,.$$

In figure 2.3 we show 2 unstable linear systems ($E^u = \mathbf{R}^2$) with corresponding Jordan normal forms

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \alpha > 0\,, \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1 > 0, \lambda_2 > 0\,.$$

and in figure 2.4 the 2 unstable linear systems ($E^s = \mathbf{R}^2$) with one eigenvalue and Jordan normal forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda > 0\,, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda > 0\,.$$

In figure 2.5 we show a center (stable but not asymptotically stable) and an hyperbolic linear system in $\mathbf{R}^2$ (unstable) with corresponding Jordan normal forms

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \alpha > 0\,, \qquad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1 < 0 < \lambda_2\,.$$

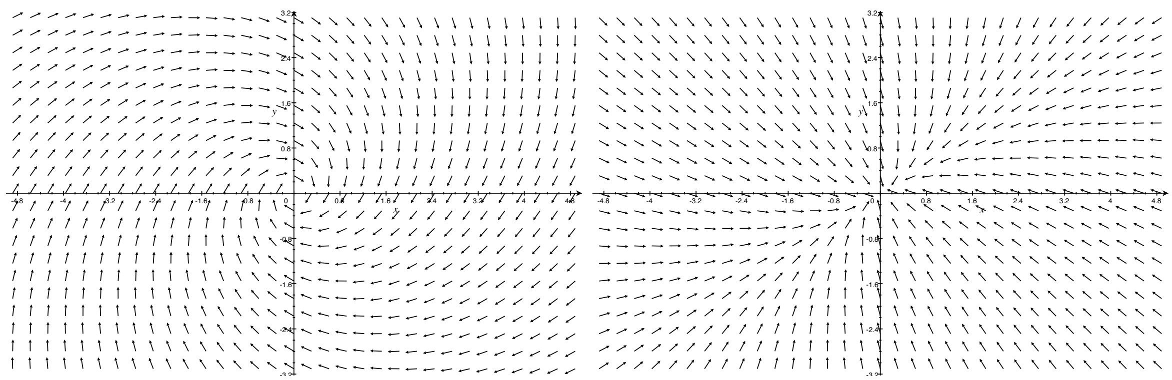

Figure 2.1: Stable linear systems with $E^s = \mathbf{R}^2$: stable spiral and stable focus with two distinct eigenvalues, stable focus with a nontrivial Jordan block

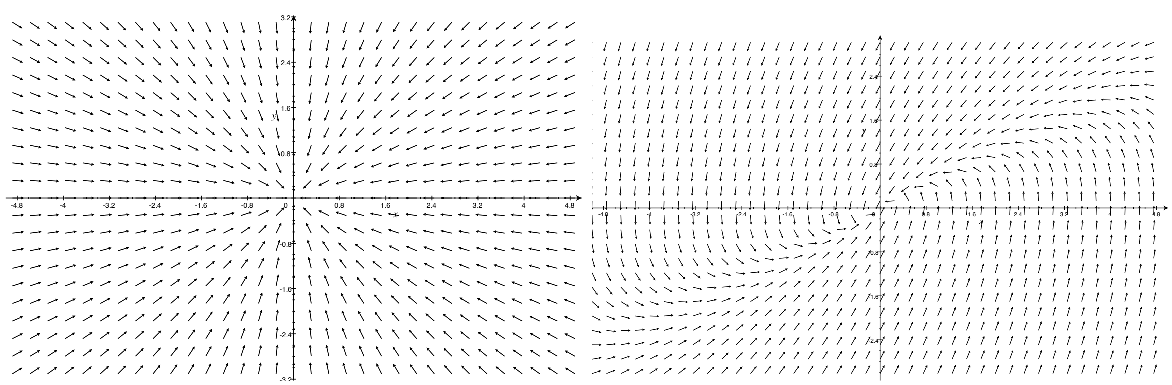

Figure 2.2: Stable linear systems with $E^s = \mathbf{R}^2$ with two identical eigenvalues: geometric multiplicity two and one.

If $A(t)$ depends on t, in general it is not enough to look at the eigenvalues of A. One can construct examples of matrices $A(t)$ whose eigenvalues are negative but for which 0 is unstable (see homework). One needs stronger condition. As an example we prove

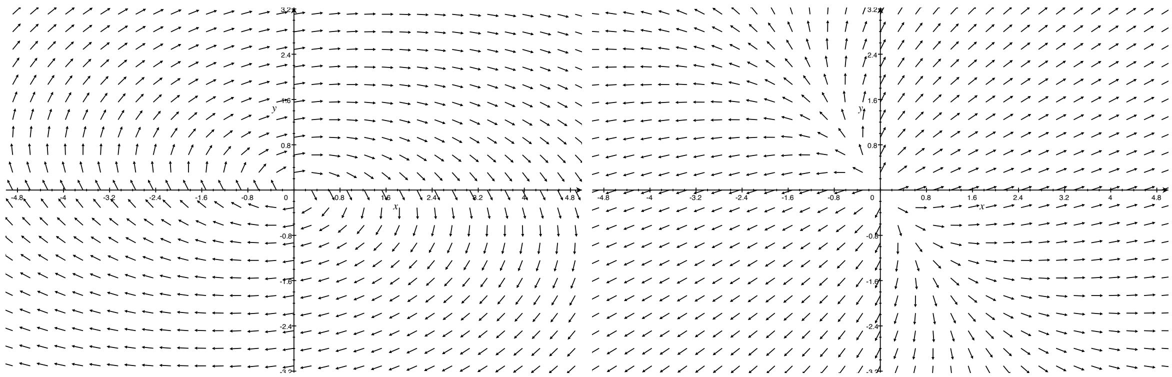

Figure 2.3: Unstable linear systems with $E^u = \mathbf{R}^2$: unstable spiral and unstable focus with two distinct eigenvalues

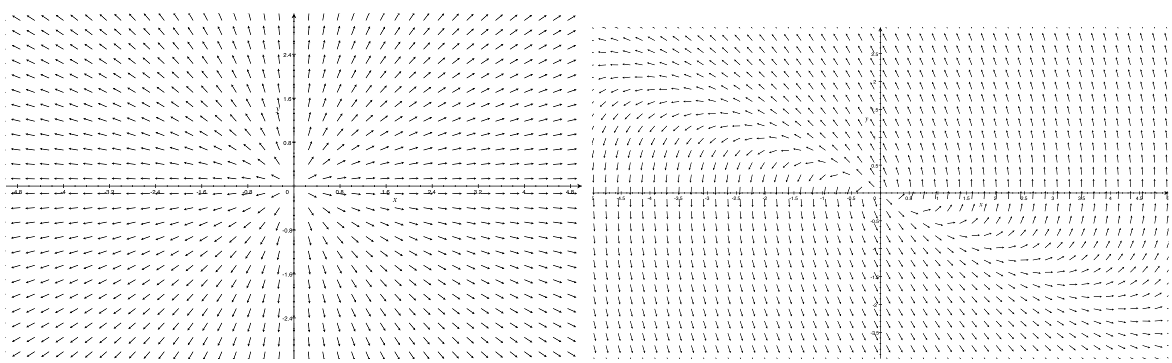

Figure 2.4: Unstable linear systems with $E^s = \mathbf{R}^2$ with two identical eigenvalues: geometric multiplicity two and one.

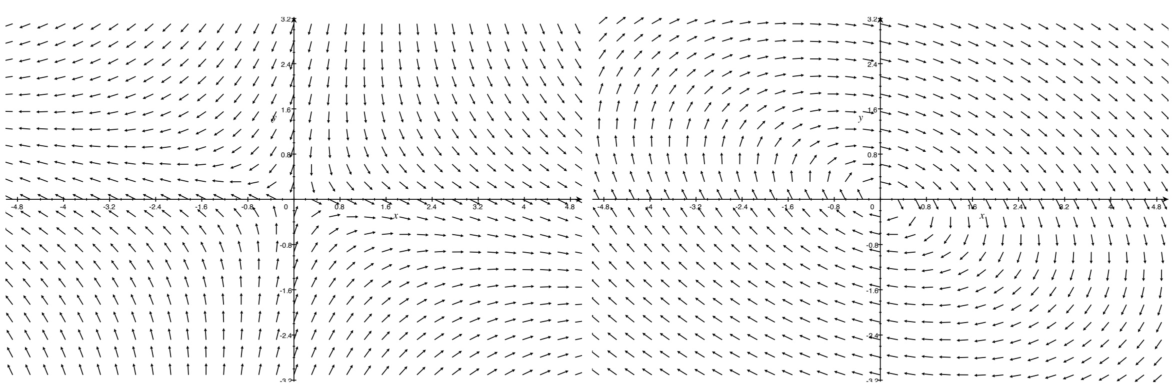

Figure 2.5: A linear hyperbolic system with $E^u \oplus E^s = \mathbf{R}^2$ and a center with $E^c = \mathbf{R}^2$.

Theorem 2.4.7 *Let $A(t)$ be symmetric, i.e., $A^*(t) = A(t)$ and continuous on $[t_0, \infty)$. If the eigenvalues $\lambda_i(t)$ of $A(t)$ satisfy $\lambda_i(t) \le \alpha$ for $t \in [t_0, \infty)$, then the solution $x(t)$ of $x' = A(t)x$ satisfy*

$$\|x(t)\|_2 \le e^{\alpha(t-t_0)} \|x(t_0)\|_2 , \quad t > t_0 . \tag{2.96}$$

In particular, if $\alpha \le 0$, then 0 is stable and if $\alpha < 0$ then 0 is asymptotically stable.

Proof: Since $A(t)$ is symmetric, its eigenvalues are real and it is diagonalizable with an orthogonal matrix: there exists a matrix $Q(t)$ with $Q^T(t) = Q^{-1}(t))$ such that $Q^T(t)A(t)Q(t) = \mathrm{diag}(\lambda_1(t), \cdots, \lambda_n(t))$. We show that, for all v and all $t > t_0$ we have

$$\langle v , Av \rangle \le \alpha \langle v , v \rangle . \tag{2.97}$$

We set $v = Qw$ and then we have $\langle v , v \rangle = \langle w , w \rangle$ and

$$\langle v , Av \rangle = \langle w , Q^T AQw \rangle \le \alpha \langle w , w \rangle = \alpha \langle v , v \rangle . \tag{2.98}$$

For a solution $x(t)$ of $x' = A(t)x$ we obtain

$$\frac{d}{dt} \|x(t)\|_2^2 = 2\langle x(t) , A(t)x(t) \rangle \le 2\alpha \|x(t)\|^2 . \tag{2.99}$$

Integrating gives

$$\|x(t)\|_2^2 \le \|x(t_0)\|_2^2 + 2\alpha \int_{t_0}^{t} \|x(s)\|_2^2 \, ds , \tag{2.100}$$

and therefore, by Gronwall Lemma,

$$\|x(t)\|_2^2 \le \|x(t_0)\|_2^2 \, e^{2\alpha(t-t_0)} . \tag{2.101}$$

■

2.5 Floquet theory

In this section we consider periodic $A(t)$, i.e., there exists $p > 0$ such that $A(t + p) = A(t)$ for all $t \in \mathbf{R}$. Such equation can be reduced, at least in principle, to the case of constants coefficients and is this reduction goes under the name of Floquet theory.

As a preliminary we need to define the logarithm of matrix. For this it is necessary to consider complex-matrix since the logarithm of a real matrix will be, in general, complex.

Proposition 2.5.1 (Logarithm of a matrix) *Let C be an invertible matrix, then there exists a (complex) matrix R such that*

$$C = e^R . \tag{2.102}$$

Proof: We will use the decomposition of $C = S + N$ into a semi-simple (i.e., diagonalizable) matrix S and a nilpotent matrix N (i.e., $N^k = 0$ for some $k \geq 1$) with $SN = NS$.
(a) Let us first consider the case $C = S$, i.e., C is semi-simple. Since S is invertible and semi-simple there exists P such that $S = P\Lambda P^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ and $\lambda_k \neq 0$ for all k. We set

$$T = PLP^{-1}\,, \quad L = \mathrm{diag}(\log\lambda_1, \cdots, \log\lambda_n)\,. \tag{2.103}$$

Then we have

$$e^T \;=\; e^{PLP^{-1}} \;=\; Pe^L P^{-1} \;=\; P\Lambda P^{-1} \;=\; S\,. \tag{2.104}$$

(b) To treat the general case $C = S + N$ we note that S is invertible if and only if C is invertible (they have the same eigenvalues) and if $SN = NS$ then $S^{-1}N = NS^{-1}$. Then we have

$$C = S + N \;=\; S(I + S^{-1}N) \tag{2.105}$$

and $S^{-1}N$ is nilpotent and commute with S.

Recall the power series

$$\log(1+t) \;=\; t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots \;=\; \sum_{j=1}^{\infty} \frac{(-1)^{j+1}t^j}{j}\,, \qquad \text{for } |t| < 1 \tag{2.106}$$

and that the formal rearrangement of power series

$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}t^j}{j} \right)^n \frac{1}{n!} \;=\; 1 + t \tag{2.107}$$

is valid for $|t| \leq 1$ (this is simply a complicated way to write the identity $e^{\log(1+t)} = 1+t$). Let us define now

$$R = T + Q \tag{2.108}$$

where T is defined as in (a) and

$$Q \;=\; \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(S^{-1}N)^j}{j}\,. \tag{2.109}$$

Since $S^{-1}N$ is nilpotent the series defining Q is actually a *finite sum* and we do need to worry about convergence. From the formal rearrangement (2.107) we conclude that

$$e^Q \;=\; (I + \lambda^{-1}N)\,. \tag{2.110}$$

Finally since T and Q commute we obtain

$$e^R = e^T e^Q = S(I + \lambda^{-1}N) = C\,, \tag{2.111}$$

and this concludes the proof of Proposition 2.5.1. ∎

Theorem 2.5.2 (Floquet) *Let $A(t)$ be a continuous periodic function of period p. Then any fundamental matrix $\Phi(t)$ for $x' = A(t)x$ has a representation of the form*

$$\Phi(t) \;=\; P(t)e^{Rt}\,, \quad P(t+p) = P(t)\,, \tag{2.112}$$

where R is a constant matrix.

Remark 2.5.3 The theorem 2.5.2 provieds us the form of the solutions. If x_0 is an eigenvector of R for the eigenvalue λ, then the solution $x(t)$ has the form $z(t)e^{\lambda t}$, where $z(t) = P(t)x_0$ is periodic with period p. More generally, by the discussion in Section 2.3, a general solution will have components which is a linear combination of terms of the form $\alpha(t)t^k e^{\lambda t}$, where $\alpha(t)$ is a vector periodic in t.

Note, in particular, that there exists periodic solutions of period p whenever 0 is an eigenvalue of R.

Proof of Theorem 2.5.2: (a) We note first that if $\Phi_1(t)$ and $\Phi_2(t)$ are two fundamental matrices, then there exists an invertible matrix C such that

$$\Phi_1(t) \;=\; \Phi_2(t)C\,. \tag{2.113}$$

This follows from the fact that

$$R(t,t_0) \;=\; \Phi_1(t)\Phi_1^{-1}(t_0) \;=\; \Phi_2(t)\Phi_2^{-1}(t_0)\,, \tag{2.114}$$

i.e.,

$$\Phi_1(t) \;=\; \Phi_2(t)\Phi_2^{-1}(t_0)\Phi_1(t_0)\,. \tag{2.115}$$

(b) If $x(t)$ is a solution of $x' = A(t)x$, then one verifies easily that $y(t) = x(t+p)$ is also a solution. Therefore if $\Phi(t)$ is a fundamental matrix, then $\Psi(t) = \Phi(t+p)$ is also a fundamental matrix. By (a) and Proposition 2.5.1, there exists a matrix R such that

$$\Phi(t+p) \;=\; \Phi(t)e^{pR}\,. \tag{2.116}$$

We now define

$$P(t) \;\equiv\; \Phi(t)e^{-tR} \tag{2.117}$$

and $P(t)$ is periodic of period p since

$$P(t+p) \;=\; \Phi(t+p)e^{-(t+p)R} \;=\; \Phi(t)e^{pR}e^{-(t+p)R} \;=\; P(t)\,. \tag{2.118}$$

This concludes the proof. ∎

The matrix $C = e^{pR}$ is called the *transition* matrix and the eigenvalues eigenvalues, λ_i, of $C = e^{pR}$ are called the *Floquet multipliers*. The matrix C depends on the choice of the fundamental matrix $\Phi(t)$, however the eigenvalues do not (see exercises).

The eigenvalues of R, μ_i are given by $\lambda_i = e^{p\mu_i}$ are called the *characteristic exponents*. They are unique, up to a multiple of $2\pi i/p$.

Remark 2.5.4 For the equation $x' = A(t)x$, let us consider the transformation

$$x(t) = P(t)y(t) \tag{2.119}$$

where $P(t)$ is the periodic matrix given by Floquet theorem. We obtain

$$x'(t) = P'(t)y(t) + P(t)y'(t) = A(t)P(t)y(t)\,. \tag{2.120}$$

On the other hand $P(t) = \Phi(t)e^{-Rt}$, so that

$$P'(t) = \Phi'(t)e^{-Rt} - \Phi(t)e^{-Rt}R = A(t)P(t) - P(t)R\,. \tag{2.121}$$

Thus we find

$$y'(t) = Ry(t)\,. \tag{2.122}$$

The transformation $x = P(t)y$ reduces the linear equation with periodic coefficients $x' = A(t)x$ to the system with constant coefficients $y' = By$. Nevertheless there are, in general, no methods available to compute $P(t)$ or the Floquet multipliers. Each equation has to be studied for itself and entire books are devoted to quite simple looking equations. The Floquet theory is however very useful to study the stability of periodic solutions, as we shall see later.

Example 2.5.5 Let us consider the equation $x''+b(t)x'+a(t) = 0$, where $a(t)$ and $b(t)$ are periodic functions. For the fundamental solution $\Phi(t) = R(t, 0)$ we have $\Phi(0) = I$ and so by Floquet Theorem

$$\Phi(p) = C = e^{pR} = \begin{pmatrix} x_1(p) & x_2(p) \\ x_1'(p) & x_2'(p) \end{pmatrix} \tag{2.123}$$

The Floquet multipliers are given by the solutions of

$$\lambda^2 + \alpha\lambda + \beta = 0\,, \tag{2.124}$$

where

$$\alpha = -x_1(p) - x_2'(p)\,, \beta = \det(C) = det(R(p, 0)) = e^{\int_0^p \mathrm{tr}A(s)\,ds} = e^{-\int_0^p b(s)\,ds} \tag{2.125}$$

In the special case where $b(s) \equiv 0$, then the equation is $\lambda^2 + \alpha\lambda + 1 = 0$. We have then

(i) If $-2 < \alpha < 2$ then the Floquet multipliers λ and $\bar{\lambda}$ are complex conjugate with modulus 1 and therefore the solutions are bounded for all $t > 0$.

(ii) If $\alpha > 2$ or $\alpha < -2$, at least one eigenvalue of C has modulus greater than 1 and there exists solutions such that $|x(t)|^2 + |x'(t)|^2$ goes to infinity as t goes to infinity.

(iii) If $\alpha = -2$, then $\lambda = 1$ is the eigenvalue of C and therefore there exists a periodic solution of period p. If $\alpha = 2$ then $\lambda = -1$ is the eigenvalue of C and therefore there exists a periodic solution of period $2p$.

2.6 Linearization

Let us consider the solution $x(t, t_0, x_0)$ of the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$. Let $\xi \in \mathbf{R}^n$ and let us now consider the solution $x(t, t_0, x_0 + \xi)$. As we have seen in Section 1.7 the function $\xi \mapsto x(t, t_0, x_0 + \xi)$ is continuous, in fact Lipschitz continuous, provided f is continuous and satisfy a Lipschitz condition. If we assume f to be of class $\mathcal{C}^1$, it is natural to ask whether the map $\xi \mapsto x(t, t_0, x_0 + \xi)$ is of class $\mathcal{C}^1$? If this is the case we have the Taylor expansion

$$x(t, t_0, x_0 + \xi) \;=\; x(t, t_0, x_0) + \frac{\partial x}{\partial x_0}(t, t_0, x_0)\xi + o(\|\xi\|)\,, \tag{2.126}$$

where $o(\|\xi\|)$ stands for a function with $\lim_{\|\xi\| \to 0} o(\|\xi\|)/\|\xi\| = 0$. The right hand side of (2.126) without the $o(\|\xi\|)$ term is called the *linearization* around the solution $x(t, t_0, x_0)$.

To obtain an idea of the form of the derivative $\frac{\partial x}{\partial x_0}(t, t_0, x_0)$ we write the Cauchy problem as

$$\frac{\partial x}{\partial t}(t, t_0, x_0) \;=\; f(t, x(t, t_0, x_0))\,, \quad x(t_0, t_0, x_0) = x_0\,, \tag{2.127}$$

and differentiate *formally* with respect to x_0. Exchanging the derivatives with respect to t and x_0 we find

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial x_0}(t, t_0, x_0) \;=\; \frac{\partial f}{\partial x}(t, x(t, t_0, x_0))\frac{\partial x}{\partial x_0}(t, t_0, x_0)\,, \quad \frac{\partial x}{\partial x_0}(t_0, t_0, x_0) = I\,. \tag{2.128}$$

This formal calculation shows that the $n \times n$ matrix $\frac{\partial x}{\partial x_0}(t, t_0, x_0)$ is a solution of the linear equation

$$\Psi' \;=\; \frac{\partial f}{\partial x}(t, x(t, t_0, x_0))\Psi\,, \quad \Psi(t_0) = I\,. \tag{2.129}$$

This equation is called the *variational equation* for the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$. It is a linear equation of the form $y' = A(t; t_0, x_0)y$, where the matrix A depends on the parameters (t_0, x_0). The resolvent also depends on this parameters and let us denote it by $R(t, s; t_0, x_0)$. The formal calculation shows that

$$\frac{\partial x}{\partial x_0}(t, t_0, x_0) \;=\; R(t, t_0; t_0, x_0)\,. \tag{2.130}$$

Before we prove that this formal computation is actually correct let us consider a number of important special cases and example

Example 2.6.1 (Linearization around equilibrium solutions) Let us assume that the ODE is autonomous, $f(t, x) = f(x)$ with f of class $\mathcal{C}^1$ and that a is a critical point, i.e., $f(a) = 0$. The constant solution $x(t,0,a) = a$ is a solution. The variational equation is then

$$\Psi' = \frac{df}{dx}(a)\Psi\,, \quad \Psi(0) = I\,, \tag{2.131}$$

whose solution is e^{tA} where $A = \frac{df}{dx}(a)$. Therefore, for small ξ, we have

$$x(t,0,a+\xi) = a + e^{tA}\xi + o(\|\xi\|)\,. \tag{2.132}$$

Example 2.6.2 Consider the mathematical pendulum $x'' + \sin(x) = 0$ or

$$\begin{aligned} x' &= y\,, \\ y' &= -\sin(x)\,. \end{aligned} \tag{2.133}$$

There are two equilibrium solution $a = (\pi, 0)^T$ and $b = (0,0)^T$ (the first component modulo 2π). The linearization around a and b gives, for small ξ,

$$z(t,0,(\pi,0)^T+\xi) \approx \begin{pmatrix} \pi \\ 0 \end{pmatrix} + e^{At}\xi\,, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{2.134}$$

where A has eigenvalues ± 1 and

$$z(t,0,\xi) \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} + e^{Bt}\xi\,, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.135}$$

where B has eigenvalues $\pm i$.

Example 2.6.3 (Linearization around a periodic solution) Let us assume that the nonlinear equation $x' = f(x)$ has a periodic solution $x(t,0,x_0) = \phi(t)$. If we linearize around this periodic solution the variational equation is given by

$$x' = \frac{df}{dx}(\phi(t))x\,. \tag{2.136}$$

which is a linear equation with periodic coefficients that we can analyze using Floquet theory of Section 2.5.

An important fact is the following. If $\phi(t)$ is aperiodic solution then $\phi'(t)$ is a solution of the variational equation (2.136). Indeed we have

$$\phi''(t) = \frac{d}{dt}f(\phi(t)) = \frac{df}{dx}(\phi(t))\phi'(t)\,. \tag{2.137}$$

In particular, in two dimensions, if $\phi(t)$ is known explicitly, this can be used to solve the variational equation, using D'Alembert reduction method (see exercises).

Example 2.6.4 The second order equation $x'' + f(x)x' + g(x) = 0$ has, under suitable conditions (see Chapter 4) a periodic solution $\phi(t)$. In this case the variational equation is given by

$$\begin{aligned} x' &= y\,, \\ y' &= -\left(f(\phi(t))\phi'(t) + g'(\phi(t))\right)x - f(\phi(t))y\,. \end{aligned} \tag{2.138}$$

and has the form $x'' + b(t)x' + a(t) = 0$, where $a(t)$ and $b(t)$ are periodic functions.

Example 2.6.5 The system

$$x' = -y + x(1 - x^2 - y^2)\,, \tag{2.139}$$
$$y' = x + y(1 - x^2 - y^2)\,, \tag{2.140}$$
$$z' = z\,. \tag{2.141}$$

has a periodic orbit in the x, y plane given by $(\cos(t), \sin(t), 0)^T$. This can be verified by direct computation or be deduced by choosing cylindrical coordinates (r, θ, z) and showing that the system (2.185) is equivalent to

$$r' = r(1 - r^2)\,, \tag{2.142}$$
$$\theta' = 1, \tag{2.143}$$
$$z' = z\,. \tag{2.144}$$

The linearization around the periodic orbit gives the variational equation

$$\Psi' = \begin{pmatrix} -2\cos^2(t) & -1 - 2\cos(t)\sin(t) & 0 \\ 1 - 2\cos(t)\sin(t) & -2\sin^2(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \Psi \tag{2.145}$$

Using the fact that $\phi'(t) = (-\sin(t), \cos(t), 0)$ is a solution, one can compute the solution of the variational equation (see exercises).

Theorem 2.6.6 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set, $f : U \to \mathbf{R}^n$ be continuous. Assume that $\frac{\partial f}{\partial x}(t, x)$ exists and is continuous on U. Then the solution $x(t, t_0, x_0)$ of $x' = f(x)$, $x(t_0) = x_0$ is continuously differentiable with respect to x_0 and its derivative $\frac{\partial x}{\partial x_0}(t, t_0, x_0)$ is a solution of the variational equation*

$$\Psi' = \frac{\partial f}{\partial x}(t, x(t, t_0, x_0))\Psi\,, \quad \Psi(t_0) = I\,. \tag{2.146}$$

Proof: For given (t, t_0, x_0) let us choose $[a, b] \subset I_{\max}$ such that $t, t_0 \in (a, b)$. Let $\xi \in \mathbf{R}^n$, we need to show that for fixed (t, t_0, x_0),

$$x(t, t_0, x_0 + \xi) - x(t, t_0, x_0) - R(t, t_0; t_0, x_0)\xi = o(\|\xi\|)\,, \tag{2.147}$$

where $o(\|\xi\|)/\|\xi\| \to 0$ as $\xi \to 0$. The integral equations for $x(t, t_0, x_0 + \xi)$, $x(t, t_0, x_0)$ and $R(t, t_0; t_0, x_0)\xi$ are respectively

$$\begin{aligned} x(t, t_0, x_0 + \xi) &= x_0 + \xi + \int_{t_0}^{t} f(s, x(s, t_0, x_0 + \xi))\, ds \\ x(t, t_0, x_0) &= x_0 + \int_{t_0}^{t} f(s, x(s, t_0, x_0))\, ds \\ R(t, t_0; t_0, x_0)\xi &= \xi + \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x(s, t_0, x_0))R(s, t_0; t_0, x_0)\xi\, ds \qquad (2.148) \end{aligned}$$

By Theorem 1.7.3, there exists a constant D such that $\|x(s, t_0, x_0 + \xi) - x(s, t_0, x_0)\| \leq D\|\xi\|$ for $t_0 \leq s \leq t$ provided $\|\xi\|$ is small enough and thus we have

$$o(\|x(s, t_0, x_0 + \xi) - x(s, t_0, x_0)\|) = o(\|\xi\|)\,. \qquad (2.149)$$

We use the Taylor approximation

$$f(s, z) - f(s, y) - \frac{\partial f}{\partial x}(s, y)(z - y) \;=\; o(\|z - y\|)\,, \qquad (2.150)$$

and we can take the right hand side to be uniform in (s, y) in any compact set K since f is of class $\mathcal{C}^1$ and will apply it to $z = x(s, t_0, x_0 + \xi)$ and $y = x(s, t_0, x_0)$ with $t_0 \leq s \leq t$.

Using these estimates with the integral equations we obtain that

$$\begin{aligned} &\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0) - R(t, t_0; t_0, x_0)\xi\| \\ &\leq \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x(s, t_0, x_0))(x(s, t_0, x_0 + \xi) - x(s, t_0, x_0) - R(s, t_0; t_0, x_0)\xi)\, ds \\ &+ \int_{t_0}^{t} o(\|x(s, t_0, x_0 + \xi) - x(s, t_0, x_0)\|)\, ds\,. \\ &\leq C \int_{t_0}^{t} \|x(s, t_0, x_0 + \xi) - x(s, t_0, x_0) - R(s, t_0; t_0, x_0)\xi\|\, ds + (b - a)o(\|\xi\|)\,, \end{aligned}$$

where $C = \sup_{s\in[a,b]} \left\{ \left\| \frac{\partial f}{\partial x}(s, x(s, t_0, x_0)) \right\| \right\}$. By Gronwall Lemma we conclude that

$$\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0) - R(t, t_0; t_0, x_0)\xi\| \leq (b - a)e^{C(b-a)}o(\|\xi\|) = o(\|\xi\|)\,, \quad (2.151)$$

and this shows that the derivative exists and satisfies the variational equation. It remains to show that the derivative $\frac{\partial x}{\partial x_0}(t, t_0, x_0)$ is a continuous function. We cannot apply Theorem 1.7.3 directly, since (t_0, x_0) are not the initial conditions for the variational equation, but are parameters of the equation. We will show this in Lemma 2.6.7.
■

Lemma 2.6.7 *Let I is an open interval and V an open set in $\mathbf{R}^q$. Assume that $A(t;c)$ is continuous on $I \times V$. Then the resolvent $R(t,t_0;c)$ for the differential equation $x' = A(t;c)x$ is a continuous function of c.*

Proof: The proof is a special case of the continuous dependence of solutions on parameters (see exercises).

Note that if the ODE is autonomous we obtain

Corollary 2.6.8 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be of class $\mathcal{C}^1$ and let us assume that the solutions of $x' = f(x)$ exist for all time. Then, for any t, the maps $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$ are of class $\mathcal{C}^1$ and so ϕ^t defines a $\mathcal{C}^1$ dynamical system, i.e. a group of diffeomorphisms.*

We discuss next the smooth dependence with respect to parameters and with respect to t_0. Let us consider a Cauchy problem $x' = f(t,x,c)$ where $f : U \to \mathbf{R}^n$ is differentiable with respect to x and c (U is an open set of $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^q$). The solution is denoted $x(t,t_0,x_0,c)$

In order to study the differentiability with respect to the parameters c we consider the extended system

$$\begin{pmatrix} x \\ c \end{pmatrix}' = \begin{pmatrix} f(t,x,c) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ c \end{pmatrix}(t_0) = \begin{pmatrix} x_0 \\ c \end{pmatrix} \tag{2.152}$$

If we set $z = (x,c)^T$ and $F(t,z) = (f(t,x,c),0)^T$, then this system becomes $z' = F(t,z)$, $z(0) = (x_0,c)^T$ and c appears only in the initial condition. Therefore we can apply Theorem 2.6.6. The function $z(t,t_0,z_0)$ is continuously differentiable with respect to $z_0 = (x_0,c)$ and therefore $x(t,t_0,x_0,c)$ is continuously differentiable with respect to c. By deriving the equation

$$\frac{\partial x}{\partial t}(t,t_0,x_0,c) \;=\; f(t,x(t,t_0,x_0,c),c)\,, \quad x(t_0,t_0,x_0,c) \;=\; x_0 \tag{2.153}$$

with respect to c we find a linear inhomogeneous equation for $\Psi(t) = \frac{\partial x}{\partial c}(t,t_0,x_0,c)$:

$$\Psi' \;=\; \frac{\partial f}{\partial x}(t,x(t,t_0,x_0,c),c)\Psi + \frac{\partial f}{\partial c}(t,x(t,t_0,x_0,c),c)\,, \quad \Psi(0) = 0\,. \tag{2.154}$$

Example 2.6.9 The solution of the problem

$$x' \;=\; f(t,x) + \epsilon g(t,x)\,, \quad x(t_0) = x_0\,. \tag{2.155}$$

is given, for small $|\epsilon|$ by $x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + o(\epsilon)$ where

$$\begin{aligned} x_0'(t) &= f(t,x_0(t))\,, \quad x(t) \;=\; x_0\,, \\ x_1'(t) &= \frac{\partial f}{\partial x}(t,x_0(t))x_1(t) + g(t,x_0(t))\,, \quad x_1(t) \;=\; 0\,. \end{aligned} \tag{2.156}$$

If we solve the first equation and find $x_0(t)$, the second equation is a linear inhomogeneous equation for $x_1(t)$

Example 2.6.10 Consider the equation $x' + x - \epsilon x^3$ with $x(0) = 1$. For $\epsilon = 0$ the solution is $x_0(t) = e^{-t}$. Expanding around this solution we find that $x(t) = x_0(t) + \epsilon x_1(t) + o(\epsilon)$ where $x_1(t)$ is a solution of

$$x_1' = -x_1 + e^{-3t}, \quad x_1(0) = 0 \tag{2.157}$$

so that $x(t) = e^{-t} + \epsilon(e^{-t} - \frac{1}{2}e^{-3t}) + o(\epsilon)$.

Let us assume that $f(t,x)$ is continuously differentiable with respect to t and x. Let us consider the extended system

$$\left(\begin{array}{c} x \\ t \end{array}\right)' = \left(\begin{array}{c} f(t,x) \\ 1 \end{array}\right), \quad \left(\begin{array}{c} x \\ t \end{array}\right)(t_0) = \left(\begin{array}{c} x_0 \\ t_0 \end{array}\right) \tag{2.158}$$

If we set $z = (x,t)^T$ and $F(z) = (f(t,x),1)^T$, then this system becomes $z' = F(z)$, $z(t_0) = (x_0,t_0)^T$ and the dependence on t_0 can be studied as the dependence on x_0. By Theorem 2.6.6, the solution $x(t,t_0,x_0)$ is continuously differentiable with respect to t_0. Differentiating the equation $\frac{\partial x}{\partial t}(t,t_0,x_0) = f(t,x(t,t_0,x_0))$ with $x(t_0,t_0,x_0,) = x_0$ with respect to t_0, one finds a linear equation for $\Psi(t) = \frac{\partial x_0}{\partial t}(t,t_0,x_0)$

$$\Psi' = \frac{\partial f}{\partial x}(t,x(t,t_0,x_0))\Psi, \quad \Psi(t_0) = -f(t_0,x_0). \tag{2.159}$$

This is the same differential equation as in the variational equation for $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ but with a different initial condition. Since $\frac{\partial x}{\partial x_0}(t,t_0,x_0)$ is the resolvent of (2.159) we obtain the relation

$$\frac{\partial x}{\partial t_0}(t,t_0,x_0) = -\frac{\partial x}{\partial x_0}(t,t_0,x_0)f(t_0,x_0). \tag{2.160}$$

We can summarize the result of this section by

Theorem 2.6.11 *Let $U \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and let $f : U \to \mathbf{R}^n$ be of class $\mathcal{C}^k$, then the solution $x(t,t_0,x_0)$ is of class $\mathcal{C}^k$ in the variables (t,t_0,x_0).*

Proof: We have proved that $x(t,t_0,x_0)$ is differentiable with respect to x_0 and t_0. The differentiability with respect to t is automatic. We can apply this argument iteratively to the variational equation and see by recurrence that x is of class $\mathcal{C}^k$.

2.7 Exercises

1. If $A \in \mathcal{L}(\mathbf{K}^n)$ the *spectral radius of* A, $\rho(A)$, is defined by

$$\rho(A) = \max\{|\lambda|\,;\,\lambda \text{ eigenvalue of } A\}. \tag{2.161}$$

(a) Let

$$A = \begin{pmatrix} 0.999 & 1000 \\ 0 & 0.999 \end{pmatrix} \tag{2.162}$$

Compute the spectral radius of A as well as $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$. Find a norm on $\mathbf{R}^n$ such that $\|A\| \le 1$.

(b) Show that for any norm on $\mathbf{K}^n$ we have the inequality $\rho(A) \le \|A\|$.

(c) Show that if A is symmetric ($A^* = A$) then we have the equality $\|A\|_2 = \rho(A)$.

(d) Show that for any A and any $\epsilon > 0$, there exists a norm such that $\|A\| \le \rho(A) + \epsilon$. *Hint:* You may use (without proof) the fact that there exists a matrix D such that DAD^{-1} is upper triangular (or maybe even in Jordan normal form). Consider the diagonal matrix S with entries $1, \mu^{-1}, \cdots, \mu^{1-n}$. Set $\|x\|_\mu = \|SDx\|$ where $\|\cdot\|$ is any norm on $\mathbf{R}^n$.

2. We have shown in class, using the binomial theorem, that if the matrices A and B commute then $e^{A+B} = e^A e^B$. Here you will show, using a different method based on uniqueness of solutions for ODE that $e^{A+B} = e^A e^B$ if and only if $AB = BA$.

   (a) Let $F(t) = Be^{tA}$ and $G(t) = e^{tA}B$. Show that if A and B commute then $F(t)$ and $G(t)$ satisfies the same ODE and thus must be equal.

   (b) Let $\Phi(t) = e^{tA}e^{tB}$ and $\Psi(t) = e^{t(A+B)}$. Show that if A and B commute, then $\Phi(t)$ and $\Psi(t)$ satisfies the same ODE and thus must be equal.

   (c) Show that if $\Phi(t) = \Psi(t)$ then A and B commute.

3. Show that if $A(t)$ is antisymmetric, i.e., $A^T = -A$, then the resolvant of $x' = A(t)x$ is orthogonal. *Hint:* Show that the scalar product of two solutions is constant.

4. **(D'Alembert reduction method)**. Consider the ODE $x' = A(t)x$ where $A(t)$ is a $n \times n$ matrix and assume that we know one non-trivial solution $x(t)$. Show that one can reduce the equation $x' = A(t)x$ to the problem $z' = B(t)z$ where $z \in \mathbf{R}^{n-1}$ and $B(t)$ is a $(n-1)\times(n-1)$ matrix. *Hint:* Without loss of generality you may assume that the n^{th} component of $x(t)$, $x_n(t) \ne 0$. Look for solutions of the form $y(t) = \phi(t)x(t) + z(t)$, where $\phi(t)$ is a scalar function and z has the form $z = (z_1, \cdots, z_{n-1}, 0)^T$.

5. (a) Using the previous problem, compute the resolvent $R(t, 1)$ of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x\,, \tag{2.163}$$

using the fact that $x(t) = (t^2, -t)^T$ is a solution. *Hint:* The solution is

$$\begin{pmatrix} t^2(1-\log t) & -t^2 \log t \\ t \log t & t(1+\log t) \end{pmatrix} \tag{2.164}$$

(b) Compute the solution of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x + \begin{pmatrix} t \\ -t^2 \end{pmatrix}, \tag{2.165}$$

with initial condition $x(1) = (0,0)^T$.

6. Compute the resolvant e^{At} for the equations $x' = Ax$ with

(a)

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}. \tag{2.166}$$

(b)

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.167}$$

(c)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}. \tag{2.168}$$

(d)

$$A = \frac{1}{9}\begin{pmatrix} 14 & 4 & 2 \\ -2 & 20 & 1 \\ -4 & 4 & 20 \end{pmatrix} \tag{2.169}$$

Hint: All eigenvalues are equal to 2.

7. The equation of motion of two coupled harmonic oscillators is

$$\begin{aligned} x_1'' &= -\alpha x_1 - \kappa(x_1 - x_2), \\ x_2'' &= -\alpha x_2 - \kappa(x_2 - x_1). \end{aligned} \tag{2.170}$$

This system is a Hamiltonian system. Find the Hamiltonian function. Find a fundamental matrix for this system. You can either write it as a first order system and compute the chararcteristic polynomial or, better, stare at the equation long enough until you make a clever Ansatz. Discuss the solution in the case where $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = 0$.

8. Consider the linear differential equation

$$x' = A(t)x\,, \quad A(t) = S(t)^{-1}BS(t) \tag{2.171}$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} \tag{2.172}$$

   (a) Show that, for any t, all eigenvalues of $A(t)$ have a negative real part.
   (b) Show, that for a suitable choice of a, the differential equation (2.171) has solutions $x(t)$ which satisfy $\lim_{t\to\infty}\|x(t)\| = \infty$. *Hint*: Set $y(t) = B(t)x(t)$.

9. Consider the system

$$x' = A(t)x\,, \qquad A(t) = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}. \tag{2.173}$$

   (a) Compute the resolvent of (2.173).
   (b) Show that $R(t,t_0) \neq \exp\left(\int_{t_0}^t A(s)\,ds\right)$.
   (c) Show that $A(t)$ does not commute with $\int_{t_0}^t A(s)\,ds$.
   (d) Show that if $A(t)$ does commute with $\int_{t_0}^t A(s)\,ds$ then the resolvent for $x' = A(t)x$ is $R(t,t_0) = \exp\left(\int_{t_0}^t A(s)\,ds\right)$

10. (a) Consider the linear inhomogenous equation $x' = Ax + f(t)$ where $f(t)$ is periodic with period p. Show that the system has a unique solution $x_p(t)$ of period p if A has no eigenvalue which is a multiple $2i\pi/p$.
    Hnt: Use Duhamel's formula to reduce the existence of a solution to the equation $x_0 = e^{pA}x_0 + W$ for the initial condition x_0.
    (b) Consider the second order equation $x'' + bx + kx = g(t)$ with $b \geq 0$ and $k \geq 0$ and $g(t)$ periodic of period p. Determine for which values of b and k the equation has a periodic solution of period p.
    (c) Show that if all the eigenvalues of A have negative real part then every solution $y(t)$ of $x' = Ax + f(t)$ converges to periodic solution found in A, i.e. $\lim_{t\to\infty} y(t) - x_p(t) = 0$. *Hint:* Use Duhamel's formula.

11. Consider the scalar equation (i.e. $n = 1$) $x' = f(t)x$ where $f(t)$ is continuous and periodic of period p.

    (a) Determine $P(t)$ and R in the decomposition of the resolvent given by Floquet Theorem.

(b) Give necessary and sufficient conditions in terms of $f(t)$ for the solutions to be bounded as $t \to \pm\infty$ or to be periodic.

12. Compute the resolvant $R(t,0)$ (in real representation) for the ODE

$$\begin{aligned} x' &= \cos(t)x - \sin(t)y\,, \\ y' &= \sin(t)x + \cos(t)y\,. \end{aligned} \tag{2.174}$$

and determine $P(t)$ and R in Floquet Theorem *Hint*: Find an equation for $z = x + iy$.

13. Consider the equation $x' = A(t)x$ where A is periodic of period p. Show that a solution $x(t)$ is asymptotically stable if and only if the Floquet multipliers have absolute value less than 1.

14. Consider the differential equation $x'' + \epsilon f(t)x = 0$, where $f(t)$ is periodic of period 2π and

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < \pi \\ 0 & \text{if} \quad \pi < t \le 2\pi \end{cases} \,. \tag{2.175}$$

For both $\epsilon = 1/4$ and $\epsilon = 4$

(a) Consider the fundamental solution $\Phi(t)$ which satisfies $\Phi(0) = \mathbf{I}$ and compute the corresponding transition matrix $C = e^{pR}$.

(b) Compute the Floquet multipliers (the eigenvalues of C).

(c) Describe the behavior of solution.

15. Let $A(t)$ be periodic of period p and consider ODE $x' = A(t)x$.

(a) Show that the transition matrix C depends on the fundamental solution, but that the eigenvalues of $C = e^{pR}$ are independent of this choice.

(b) Show that for each Floquet multiplier λ (the eigenvalue of C), there exists a solution of $x' = A(t)x$ such that $x(t+p) = \lambda x(t)$, for all t.

16. Consider the equation $x' = A(t)x$ where $A(t)$ is periodic of period p.

(a) Use Floquet Theorem and Liouville Theorem to show that

$$\det(e^{pR}) = e^{\int_0^p \mathrm{Trace}(A(s))\,ds}\,. \tag{2.176}$$

(b) Deduce from (a) that the characteristic exponents μ_i satisfy

$$\mu_1 + \cdots + \mu_n = \frac{1}{p}\int_0^p \mathrm{Trace}(A(s))\,ds \tag{2.177}$$

17. Show that the system

$$x' = -y + x(1 - x^2 - y^2)\,, \tag{2.178}$$
$$y' = x + y(1 - x^2 - y^2)\,, \tag{2.179}$$
$$z' = z\,. \tag{2.180}$$

is expressed in cylinder coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ by

$$r' = r(1 - r^2)\,, \tag{2.181}$$
$$\theta' = 1, \tag{2.182}$$
$$z' = z\,. \tag{2.183}$$

It is easy to verify that $\phi(t) = (-\sin(t), \cos(t), 0)$ is a periodic orbit. Determine the corresponding variational equation$\Psi' = A(t)\Psi = f'(\phi(t))\Psi$ and solve it.

18. Consider the equation for the mathematical pendulum

$$x'' + \sin(x) = 0\,, \quad x(0) = \epsilon\,, x'(0) = 0\,, \tag{2.184}$$

where ϵ is supposed to be small. Show that the solution can be written in the form

$$x(t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + O(\epsilon^4)\,. \tag{2.185}$$

Compute $x_1(t)$, $x_2(t)$, and $x_3(t)$. *Hint:* Taylor expansion.

Chapter 3

Stability analysis

3.1 Stability of critical points of nonlinear systems

Consider the autonomous equation $x' = f(x)$ and let us assume that a is an *isolated* critical point, i.e. $f(a) = 0$ and there exists a neighborhood $B_r(a)$ of a such that $B_r(a)$ contains no other singularities of a. We will study the behavior of solutions in a neighborhood of a.

It is convenient to change variable and set $y = x - a$ and define $g(y) \equiv f(y + a)$ so that $g(0) = 0$. Then we have $y' = x' = f(x) = f(y + a) = g(y)$. Therefore we can and will always assume that the critical point is $a = 0$.

If f is of class $\mathcal{C}^1$, we can linearize around 0. We write $f(x)$ as

$$f(x) = \frac{df}{dx}(0)x + g(x) \quad \text{or} \quad g(x) = f(x) - \frac{df}{dx}(0)x\,. \tag{3.1}$$

We have $g(0) = 0$ and

$$g(x) \;=\; g(x) - g(0) \;=\; \int_0^1 \frac{d}{ds} g(sx)\,ds \;=\; \int_0^1 \left(\frac{df}{dx}(sx) - \frac{df}{dx}(0)\right) x\,ds \tag{3.2}$$

and thus

$$\|g(x)\| \;=\; \sup_{\{y\,;\,\|y\|\le\|x\|\}} \left\|\frac{df}{dx}(y) - \frac{df}{dx}(0)\right\| \|x\|\,, \tag{3.3}$$

and so

$$\lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} \;=\; 0\,. \tag{3.4}$$

So the differential equation has the form

$$x' \;=\; Ax + g(x) \tag{3.5}$$

where $A = \frac{df}{dx}(0)$ and g satisfies (3.4).

Theorem 3.1.1 (Stability Theorem) *Let $g : (t_0, \infty) \times U$ be continuous and locally Lipschitz in U where U is a neighborhood of 0. Let us assume that*

$$\lim_{\|x\| \to 0} \sup_{t > t_0} \frac{\|g(t,x)\|}{\|x\|} \,=\, 0\,. \tag{3.6}$$

Let A be a $n \times n$ matrix whose all eigenvalues have negative real part, $\mathrm{Re}\lambda_i < 0$. Then the zero solution of

$$x' \,=\, Ax + g(t,x) \tag{3.7}$$

is asymptotically stable.

Proof: Since g continuous and locally Lipschitz, we have existence of solutions $x(t) = x(t, t_0, x_0)$ if x_0 is in a neighborhood of 0. We use the following generalization of Duhamel's formula: $x(t)$ is solution of the integral equation

$$x(t) \,=\, e^{At}x_0 + \int_{t_0}^{t} e^{A(t-s)} g(x(s))\, ds\,. \tag{3.8}$$

One can verify this formula by differentiation. Since the real parts of the eigenvalues of A are negative we conclude that there exists constants $K > 0$ and $\mu > 0$ such that

$$\|e^{A(t-t_0)}\| \,\le\, Ke^{-\mu(t-t_0)}\,. \tag{3.9}$$

From this we deduce the estimate

$$\|x(t)\| \,\le\, Ke^{-\mu(t-t_0)}\|x_0\| + K\int_{t_0}^{t} e^{-\mu(t-s)}\|g(x(s))\|\, ds\,. \tag{3.10}$$

Since $\|g(t,x)\|/\|x\| \to 0$ uniformly in t, for any $b > 0$ there exists $\epsilon > 0$ such that $\|g(t,x)\| \le b\|x\|$ provided $\|x\| \le \epsilon$. In the sequel we choose $b = \frac{\mu}{2K}$.

As long as the solution $x(t)$ stays in $\{x\,;\, \|x\| \le \epsilon\}$ we have the bound

$$e^{\mu(t-t_0)}\|x(t)\| \,\le\, K\|x_0\| + Kb\int_{t_0}^{t} e^{\mu(s-t_0)}\|x(s)\|\, ds\,. \tag{3.11}$$

By Gronwall Lemma we obtain

$$e^{\mu(t-t_0)}\|x(t)\| \,\le\, K\|x_0\|e^{bK(t-t_0)}\,. \tag{3.12}$$

or, with $b = \frac{\mu}{2K}$,

$$\|x(t)\| \,\le\, K\|x_0\|e^{-\frac{\mu}{2}t}\,. \tag{3.13}$$

We set $\delta := \frac{\epsilon}{K}$. If $\|x_0\| \le \delta$, then the estimate shows that $x(t)$ stays in $\{x\,;\, \|x\| \le \epsilon\}$ for all $t > 0$. This shows that the zero solution is asymptotically stable. ∎

We prove next an instability result. We will consider the equation $x' = Ax + g(t,x)$ and assume that A has at least one eigenvalue has a positive real part.

Theorem 3.1.2 (Instability Theorem) *Let $g : (t_0, \infty) \times U$ be continuous and locally Lipschitz in U where U is a neighborhood of 0. Let us assume that*

$$\lim_{\|x\| \to 0} \sup_{t > t_0} \frac{\|g(t,x)\|}{\|x\|} = 0 \,. \tag{3.14}$$

Let A be a $n \times n$ matrix and let us suppose that at least one eigenvalue of A has a positive real part. Then the zero solution of

$$x' = Ax + g(t,x) \tag{3.15}$$

is unstable.

Proof: We first transform the differential equation into a form which is better suited to our purposes. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A (counting multiplicities). There exists an invertible matrix S such that $B = S^{-1}AS$ is in Jordan normal form, i.e., $b_{ii} = \lambda_i$ and $b_{i,i+1} = 1$ or 0 and all other $b_{ij} = 0$. Let H be the diagonal matrix $H = \mathrm{diag}(\eta, \eta^2, \cdots, \eta^n)$ and so $H^{-1} = \mathrm{diag}(\eta^{-1}, \eta^{-2}, \cdots, \eta^{-n})$. It is easy to check that for the matrix $C = H^{-1}BH$ we have $c_{ii} = \lambda_i$ and $c_{i,i+1} = \eta$ or 0 and and all other $c_{ij} = 0$.

We now set $x(t) = SHy(t)$, then the equation (3.15) transforms into

$$y' = Cy + h(t,y) \,, \tag{3.16}$$

where

$$h(t,y) \equiv H^{-1}S^{-1}g(t, SHy) \,. \tag{3.17}$$

Since g satisfies condition (3.14), so does h. Indeed from $\|g(t,x)\| \le b\|x\|$ for $\|x\| \le \delta$ it follows that

$$\|h(t,y)\| \le \|H^{-1}S^{-1}\| \, \|SH\| \, b\|y\| \,, \quad \text{for } \|y\| \le \frac{\delta}{\|SH\|} \,. \tag{3.18}$$

The i^{th} component of (3.16) has either the form

$$y_i' = \lambda_i y_i + h_i(t,y) \,, \tag{3.19}$$

or

$$y_i' = \lambda_i y_i + \eta y_{i+1} + h_i(t,y) \,. \tag{3.20}$$

Let us denote by j the indices for which $\mathrm{Re}\lambda_j > 0$ and by k the indices for which $\mathrm{Re}\lambda_k \le 0$. We set

$$R(t) = \sum_j |y_j(t)|^2 \,, \quad r(t) = \sum_k |y_k(t)|^2 \,. \tag{3.21}$$

Let us choose η such that

$$0 \;<\; 6\eta \;<\; \mathrm{Re}\lambda_j\,, \qquad \text{for all } j \tag{3.22}$$

and δ so small that

$$\|h(t,y)\| \le \eta\|y\| \qquad \text{for } \|y\| \le \delta\,. \tag{3.23}$$

If $y(t)$ is a solution of (3.19) or (3.20) with

$$\|y_0\| \le \delta\,, \quad r(0) \le R(0)\,, \tag{3.24}$$

then as long as $\|y(t)\| \le \delta$ and $r(t) \le R(t)$ we have

$$\begin{aligned} R'(t) &= 2\sum_j \mathrm{Re}\, y_j'\bar{y}_j \\ &= \sum_j 2\mathrm{Re}\,\lambda_j y_j \bar{y}_j + \{\eta \mathrm{Re}\, y_{j+1}\bar{y}_j\} + \mathrm{Re}\,\bar{y}_j h_j(t,y) \end{aligned} \tag{3.25}$$

where the term in brackets appears or not depending on j. By Cauchy-Schwartz inequality we have

$$\left|\sum_j \mathrm{Re}\, y_{j+1}\bar{y}_j\right| \;\le\; \sum_j |y_{j+1}y_j| \;\le\; \sqrt{\sum_j |y_j|^2 \sum_j |y_{j+1}|^2} \;\le\; R\,. \tag{3.26}$$

and

$$\left|\sum_j \mathrm{Re}\,\bar{y}_j h_j(t,y)\right| \;\le\; \sqrt{\sum_j |y_j|^2 \sum_j |h_j|^2} \;\le\; R^{1/2}\|h\|.\,, \tag{3.27}$$

Since we assumed that $r(t) \le R(t)$ and $\|y(t)\| \le \delta$ we have

$$R^{1/2}\|h\| \;\le\; R^{1/2}\eta\|y\| \;\le\; \eta R^{1/2}\sqrt{R+r} \;\le\; 2\eta R \tag{3.28}$$

and

$$\sum_j \mathrm{Re}\,\lambda_j y_j \bar{y}_j > 6\eta R\,. \tag{3.29}$$

Therefore we have the equation

$$\frac{1}{2}R'(t) \;>\; 6\eta R - \eta R - 2\eta R \;=\; 3\eta R\,. \tag{3.30}$$

A similar equation holds for r. Using $\mathrm{Re}\lambda_k \le 0$ one obtains

$$\frac{1}{2}r'(t) \;<\; \eta r + 2\eta R\,. \tag{3.31}$$

. As long as $r(t) < R(t)$ we have

$$\frac{1}{2}(R' - r') > \eta(3R - r - 2R) = \eta(R - r) > 0, \tag{3.32}$$

i.e., the difference $R - r$ is increasing as long as it is positive. Therefore

$$\|y(t)\|^2 \geq R^2 - r^2 \geq (R^2(t_0) - r^2(t_0))e^{2\eta(t-t_0)}. \tag{3.33}$$

So this solution leaves the domain given by $\|y\| \leq \delta$, this means that the trivial solution is unstable. ∎

From Theorems 3.1.1 and 3.1.2 we obtain immediately

Corollary 3.1.3 *Let $f(x)$ be a function of class $\mathcal{C}^2$ and let a be a critical point of f, i.e. $f(a) = 0$.*

1. *If the eigenvalues of $A = \frac{df}{dx}(a)$ have all a negative real part, then the critical point a is asymptotically stable.*
2. *If at least one of the eigenvalues of $A = \frac{df}{dx}(a)$ has all a positive real part, then the critical point a is unstable.*

Example 3.1.4 The Predator-Prey equations are given by

$$x' = x(\alpha - \beta y), \quad y' = y(\gamma x - \delta), \tag{3.34}$$

where $\alpha, \beta, \gamma, \delta$ are given positive constants and we assume that $x \geq 0$ and $y \geq 0$.

$$\frac{dx}{dy} = \frac{x}{y}\frac{(\alpha - \beta y)}{(\gamma x - \delta)} \quad \text{or} \quad \frac{(\gamma x - \delta)}{x}dx = \frac{(\alpha - \beta y)}{y}dy. \tag{3.35}$$

There are two critical points $a = (0, 0)$ and $b = (\delta/\gamma, \alpha/\beta)$. The linearization around $(0, 0)$ yields

$$A = \frac{df}{dx}(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}, \tag{3.36}$$

with eigenvalues α and $-\delta$, and

$$B = \frac{df}{dx}(\delta/\gamma, \alpha/\beta) = \begin{pmatrix} 0 & -\beta\delta/\gamma \\ \gamma\alpha/\beta & 0 \end{pmatrix}, \tag{3.37}$$

with eigenvalues $\lambda = \pm i\sqrt{\alpha\delta}$. From Theorem 3.1.2 we conclude that $(0, 0)$ is unstable while neither Theorem 3.1.2 nor Theorem 3.1.1 apply to the critical point $b = (\delta/\gamma, \alpha/\beta)$ which is linearly stable.

On the other hand, as we have seen in Example 1.1.2, we know that the solutions lie on the curves

$$\gamma x - \delta \log x + \beta y - \alpha \log y \;=\; c \tag{3.38}$$

and therefore are periodic. This shows (see Figure 1.2) that $b = (\delta/\gamma, \alpha/\beta)$ is still stable for the nonlinear equation. Note also, even though $(0,0)$ is unstable, any solution which starts close to $(0,0)$ will come back close to $(0,0)$ infinitely many times.

In the previous example we have a critical point which has a stable linearization and is stable. This is by no means typical. The nonlinearity can can change a linearly stable critical point into an asymptotically stable or an unstable critical point as the following example shows.

Example 3.1.5 Consider the equation

$$\begin{aligned} x' &= y - \mu x(x^2 + y^2)\,, \\ y' &= -x - \mu y(x^2 + y^2)\,. \end{aligned} \tag{3.39}$$

The point $(0,0)$ is a critical point. The linearized system around $(0,0)$ is $x' = y$, $y' = -x$ with eigenvalues $\pm i$ and thus stable. To investigate the behavior of the nonlinear system we change into polar coordinates and find

$$\theta' = 1\,, \quad r' \;=\; -\mu r\,. \tag{3.40}$$

From this we see that if $\mu > 0$ the critical point $(0,0)$ is asymptotically stable and if $\mu < 0$ $(0,0)$ is asymptotically stable.

Example 3.1.6 (Competing species) Consider the set of equations

$$\begin{aligned} x' &= x - ax^2 - cxy\,, \quad x \geq 0\,, \\ y' &= y - by^2 + dxy \quad y \geq 0\,, \end{aligned} \tag{3.41}$$

where $a, b, c, d > 0$. This models the competition of two species living in a certain territory. If, say $y = 0$, then $y' = 0$ and $x' = x - ax^2$ (logistic equation) then the population x has linear growth rate with a natural limit ($x = 1/a$ is an asymptotically stable equilibrium). A similar situation holds if $x = 0$. This also implies that if x_0 and y_0 are nonnegative they remain so forever. The third term on the right side of (3.41) favors species y over species x if they are interacting.

The critical points with their linearization are given by

$$(0,0)\,,\ (0,1/b)\,,\ (1/a,0)\,,\ \left(\frac{b-c}{ab+cd}\,,\ \frac{a+d}{ab+cd}\right) \tag{3.42}$$

If $b \geq c$ (weak interaction) the fourth critical point is found in the domain of interest $(x, y > 0)$ while if $b < c$ (strong interaction) we have only three relevant critical points. For the first three critical points linearization gives gives

crit.point	linearization	eigenvalues	$b > c$	$b < c$
$(0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\lambda_1 = 1, \lambda_2 = 1$	source	source
$(0, \frac{1}{b})$	$\begin{pmatrix} 1 - \frac{c}{b} & 0 \\ \frac{d}{b} & -1 \end{pmatrix}$	$\lambda_1 = -1, \lambda_2 = 1 - \frac{c}{b}$	saddle	sink
$(\frac{1}{a}, 0)$	$\begin{pmatrix} -1 & -\frac{c}{a} \\ 0 & 1 + \frac{d}{a} \end{pmatrix}$	$\lambda_1 = -1, \lambda_2 = 1 + \frac{d}{a}$	saddle	saddle

(3.43)

For the last critical point linearization around $\left(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd}\right)$ gives

$$A = \frac{1}{ab+cd} \begin{pmatrix} -a(b-c) & -c(b-c) \\ d(a+d) & -b(a+d) \end{pmatrix}. \tag{3.44}$$

For $b > c$ we have $\lambda_1 \lambda_2 = \det(A) > 0$ and $\lambda_1 + \lambda_2 = \text{Trace}(A) < 0$ from which we conclude that A has 2 eigenvalues with negative real part and so we have a stable critical point (both a sink or a spiral are possible).

In the case of strong interaction $(b < c)$ the species y will die out while for weak interaction $(b > c)$ there exists a positive stable equilibrium where the two species coexist, see figure 3.1

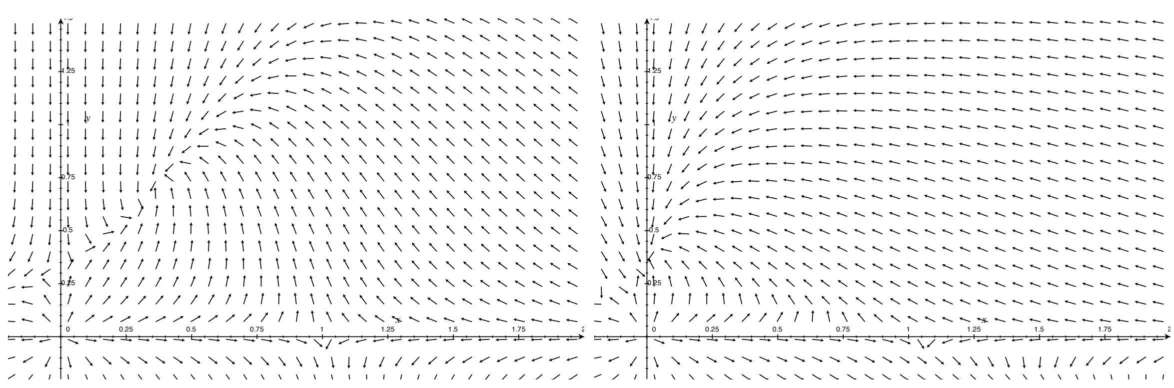

Figure 3.1: Competing species with $a = 1$, $c = 3$, $d = 3$, and $b = 1$ (left) and $b = 5$(right)

3.2 Stable and unstable manifold theorem

We continue our investigation of the behavior of solutions near a critical point of the nonlinear equation $x' = f(x)$, where f is of class $\mathcal{C}^1$. As we have seen in Section 3.1 we can always assume that the critical point is 0 and that the equation has the form

$$x' = Ax + g(x)\,, \tag{3.45}$$

and $g(x) = f(x) - \frac{df}{dx}(0)x$ satisfies $g(0) = 0$ and $\frac{dg}{dx}(0) = 0$. Moreover for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|x\| < \delta$, $\|y\| < \delta$ implies that $\|g(x) - g(y)\| \le \epsilon\|x - y\|$.

Definition 3.2.1 A critical point a for the ODE $x' = f(x)$ is called *hyperbolic* if the matrix $A = \frac{df}{dx}(a)$ has no eigenvalues with zero real part.

Let 0 be an hyperbolic critical point for the linear equation $x' = Ax$, where $x \in \mathbf{R}^n$ and A has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real parts (counting multiplicities). We can split $\mathbf{R}^n$ into invariant stable and unstable subspace, $\mathbf{R}^n = E^s \otimes E^u$ with $\dim E^s = k$ and $\dim E^u = n - k$. For any point x in the stable subspace E^s we have $\lim_{t\to\infty} \|x(t)\| = 0$ and the convergence is exponentially fast. For any point in the unstable subspace E^u we have $\lim_{t\to-\infty} \|x(t)\| = 0$ and the convergence is also exponentially fast.

We will show that the situation is essentially the same for the nonlinear equation $x' = Ax + f(x)$ in a neighborhood of an hyperbolic critical points: There exists a invariant manifold (in fact a k-dimensional hypersurface) W^s (the local stable manifold) defined locally around the origin which is tangent to E^s and such that any solution of $x' = Ax + f(x)$ which starts on W^u converges exponential fast to 0 as $t \to \infty$. Similarly there exists a manifold W^s of dimension $n - k$, tangent to E^s at the origin, such that such that any solution of $x' = Ax + f(x)$ which starts on W^s converges exponential fast to 0 as $t \to \infty$.

We can always find an invertible C such that $C^{-1}AC$ is block diagonal

$$B = C^{-1}AC = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}. \tag{3.46}$$

and the eigenvalues of the $k \times k$ matrix P have all negative real parts and the eigenvalues of the $n-k \times n-k$ matrix Q have all positive real parts. If we set $x = Cy$, the equation becomes

$$y' = By + h(y) \tag{3.47}$$

and $h(y) = C^{-1}g(Cy)$ has the same properties as g.

Theorem 3.2.2 *Stable Manifold Theorem Let A be a $n \times n$ matrix with k eigenvalues with negative real part and $n - k$ eigenvalues with positive real parts. Let $g : U \to \mathbf{R}^n$*

(U is a neighborhood of the origin) be a function of class $\mathcal{C}^1$ which satisfy $g(0) = 0$, $g'(0) = 0$, and for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|g(x) - g(y)\| \leq \epsilon \|x - y\| \,, \qquad \textit{whenever } \|x\| < \delta \,, \|y\| < \delta \,. \tag{3.48}$$

Consider the ODE

$$x' = Ax + g(x) \,. \tag{3.49}$$

Then there exists a manifold W^s (the local stable manifold) which is invariant under the flow of (3.49) and contains the origin. Any solution of (3.49) which start on W^s converges exponential fast to 0 as $t \to \infty$. Moreover there exists a manifold W^u (the local unstable manifold) which is invariant under the flow of (3.49) and contains the origin. Any solution of (3.49) which start on W^u converges exponential fast to 0 as $t \to -\infty$.

Proof: We can always find an invertible C such that $C^{-1}AC$ is block diagonal

$$B \;=\; C^{-1}AC \;=\; \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} . \tag{3.50}$$

and the eigenvalues of the $k \times k$ matrix P have all negative real parts and the eigenvalues of the $n-k \times n-k$ matrix Q have all positive real parts. If we set $x = Cy$, the equation becomes

$$y' \;=\; By + h(y) \tag{3.51}$$

and $h(y) = C^{-1}g(Cy)$ has the same condition as g does.

We will construct W^u for (3.51) as the graph of a function $\Psi : \mathbf{R}^k \to \mathbf{R}^{n-k}$. The equations

$$y_j \;=\; \Psi_j(y_1, \cdots, y_k) \,, \qquad j = k+1, \cdots, n \,. \tag{3.52}$$

define a k-dimensional hypersurface in the y-space $\mathbf{R}^n$. The unstable manifold in x-space for (3.49) is then obtained by the linear change of variables $x = Cy$.

Let us define

$$U(t) \;=\; \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} , \quad V(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} . \tag{3.53}$$

Then we have

$$e^{Bt} \;=\; U(t) + V(t) \,, \tag{3.54}$$

and

$$U'(t) \;=\; PU(t) \;=\; BU(t) \,, \qquad V'(t) \;=\; QV(t) \;=\; BV(t) \,. \tag{3.55}$$

Since all eigenvalues of P (resp Q) have negative (resp. positive) real parts, we set $\alpha = \max_{\mathrm{Re}\lambda_j < 0} \mathrm{Re}\lambda_i$ and choose σ sufficiently small such that

$$\begin{aligned} \|U(t)\| \;&\leq\; Ke^{-(\alpha+\sigma)t} \,, \quad \text{for all } t \geq 0 \,, \\ \|U(t)\| \;&\leq\; Ke^{\sigma t} \,, \quad \text{for all } t \leq 0 \,. \end{aligned} \tag{3.56}$$

Nest consider the integral equation

$$u(t,a) \,=\, U(t)a + \int_0^t U(t-s)h(u(s,a))\,ds - \int_t^\infty V(t-s)h(u(s,a))\,ds\,. \qquad (3.57)$$

By differentiating and using (3.55) one sees easily that if $u(t,a)$ is a continuous solution of the integral equation (3.57), then it is also a solution of $y' = Ay + h(y)$. We solve this integral equation by the methods of successive approximations with

$$\begin{aligned} u_0(t,a) &= 0\,, \\ u^{(m+1)}(t,a) &= U(t)a + \int_0^t U(t-s)h(u^{(m)}(s,a))\,ds - \int_t^\infty V(t-s)h(u^{(m)}(s,a))\,ds\,. \end{aligned}$$

We assume that $\epsilon \le \sigma/4k$ and $\|a\| \le \delta/2K$ where ϵ and δ are given by the Lipschitz-type condition on h. We show then by induction that

$$\begin{aligned} \|u^{(m)}(t,a)\| &\le 2K\|a\|e^{-\alpha t}\,, & (3.58) \\ \|u^{(m)}(t,a) - u^{(m-1)}(t,a)\| &\le \frac{K\|a\|e^{-\alpha t}}{2^{m-1}}\,, \quad \text{for } t \ge 0. & (3.59) \end{aligned}$$

Let us assume that (3.58) holds for $u^{(m)}(t,a)$. Then, $\|u^{(m)}(t,a)\| \le 2K\|a\| \le \delta$ and so, using that $\|h(y)\| \le \epsilon\|y\|$ if $\|y\| \le \delta$ and the bounds on $U(t)$ and $V(t)$ we have

$$\begin{aligned} &\|u^{(m+1)}(t,a)\| \\ &\le\; \|U(t)\|\|a\| + \epsilon\int_0^t \|U(t-s)\|\|u^{(m)}s,a)\|\,ds + \epsilon\int_t^\infty \|V(t-s)\|\|u^{(m)}(s,a)\|\,ds \\ &\le\; Ke^{-(\alpha+\sigma)t}\|a\| + 2\epsilon K^2\|a\|\int_0^t e^{-(\alpha+\sigma)(t-s)}e^{-\alpha s}\,ds + 2\epsilon K^2\|a\|\int_t^\infty e^{\sigma(t-s)}e^{-\alpha s}\,ds \\ &\le\; Ke^{-(\alpha+\sigma)t}\|a\| + 2\epsilon K^2 e^{-\alpha t}\frac{1}{\sigma} + 2\epsilon K^2 e^{-\alpha t}\frac{1}{\alpha+\sigma} \le 2Ke^{-\alpha t}\,, \qquad (3.60) \end{aligned}$$

where in the last inequality we have used that $\epsilon \le \sigma/4K$.

Let us consider next (3.59). It holds for $m=1$ because $h(0)=0$ and because of the bound on $\|U(t)\|$. Since $u^{(m)}(t,a) \le 2K\|a\| \le \delta$ we use that $\|h(y)-h(z)\| \le \epsilon\|y-z\|$ if $\|y\|,\|z\| \le \delta$ and we have, by the induction hypothesis,

$$\|u^{(m+1)}(t,a) - u^{(m)}(t,a)\|$$

$$\|u_n(s,a) - u_{n-1}(s,a)\|\,ds$$
$$\|u_n(s,a) - u_{n-1}(s,a)\|ds$$

$$\begin{aligned} &\le\; \epsilon\int_0^t Ke^{-(\alpha+\sigma)(t-s)}\frac{K\|a\|e^{-\alpha s}}{2^{m-1}}\,ds \;+\; \epsilon\int_t^\infty e^{\sigma(t-s)}\frac{K\|a\|e^{-\alpha s}}{2^{m-1}}\,ds \\ &\le\; \frac{\epsilon K^2\|a\|}{2^{n-1}}\left(e^{-(\alpha+\sigma)t}\int_0^t e^{\sigma s}\,ds + e^{\sigma t}\int_t^\infty e^{-(\alpha+\sigma)s}\,ds\right) \\ &\le\; \frac{\epsilon K^2\|a\|e^{-\alpha t}}{2^{m-1}}\left(\frac{1}{\sigma} + \frac{1}{\sigma+\alpha}\right) \;\le\; \frac{K\|a\|e^{-\alpha t}}{2^m}\,, \qquad (3.61) \end{aligned}$$

where, in the last inequality, we assume that $\epsilon < \sigma/4K$.

The bound (3.59) and using telescopic sum implies that for $k > m$ and $t > 0$

$$\|u^{(k)}(t,a) - u^{(m)}(t,a)\| \leq K\|a\| \sum_{j=m}^{\infty} \frac{1}{2^j}, \tag{3.62}$$

and thus $\{u^{(m)}(t,a)\}$ is a Cauchy sequence uniformly in $t > 0$. Thus $u(t,a) = \lim_{n\to\infty} u^{(m)}(t,a)$ is continuous and satisfies the integral equation (3.57). Moreover by (3.58), $u(t,a)$ satisfies the bound

$$\|u(t,a)\| \leq 2Ke^{-\alpha t}, \quad \text{for } t \geq 0 \quad \text{and } \|a\| \leq \delta/2K. \tag{3.63}$$

From the form of the integral equation (3.57), it is clear that the last $n-k$ components of the vector a do not enter the computation and hence they may and will be taken to be 0. The components of $u(t,a) = (u_1(t,a), \cdots u_n(t,a))$ satisfy the initial conditions

$$u_j(0,a) = a_j \quad \text{for} \quad j = 1, \cdots, k. \tag{3.64}$$

and

$$u_j(0,a) = -\int_0^\infty V(-s)h(u(s,a_1,\cdots,a_k,0,\cdots 0)) \quad \text{for} \quad j = k+1,\cdots,n. \tag{3.65}$$

Now we define the function $\Psi = (\psi_{j+1}, \cdots, \psi_n)$ by

$$\psi_j(a_1,\cdots,a_k) = u_j(0,a_1,\cdots,a_k). \tag{3.66}$$

The map Ψ is defined in a neighborhood of the origin and define a $n-k$ dimensional hypersurface S $y_j = \Psi_j(y_1, cdots, y_k)$.

If $y(t)$ is a solution of $y' = By + h(y)$ with $y(0) \in S$, i.e., $y(0) = u(0,a)$ then $y(t) = u(t,a)$ and by (3.63) $\lim_{t\to\infty} y(t) = 0$.

On the other hand if $y(t)$ is a solution with $\|y(0)\|$ small and $y(0)$ not on S, then $y(t)$ will exit the ball $\{\|y\| \leq \delta\}$. By contradiction suppose that $y(t) \leq \delta$ for all $t \geq 0$. Then we have

$$\begin{aligned} y(t) &= e^{Bt}y(0) + \int_0^\infty e^{B(t-s)}h(y(s))\,ds \\ &= U(t)y(0) + V(t)c + \int_0^t U(t-s)h(y(s))\,ds - \int_t^\infty V(t-s)h(y(s))\,ds \end{aligned} \tag{3.67}$$

where c is the vector

$$c = y(0) + \int_0^\infty V(-s)h(y(s))\,ds. \tag{3.68}$$

The integral in (3.68) converges by the bound on $\|V(t)\|$ and since $h(y(s))$ is bounded. Then all terms in (3.67) are uniformly bounded in t except possible the term $V(t)c$.

Since all eigenvalues of q have positive real parts, we have $\|V(t)\| \geq e^{\gamma t}$. Thus $V(t)c$ is unbounded unless $c = 0$. But if $c = 0$, $y(0) \in S$ which is a contradiction.

The existence of an unstable manifold is proved exactly in the same way by reversing time $t \mapsto -t$, i.e., by considering the system

$$y' = -By - h(y)\,. \tag{3.69}$$

The stable manifold for (3.69) is the unstable manifold for $y' = By + h(y)$. This concludes the proof of the theorem. ∎

Remark 3.2.3 With a little more work one can show that the function Ψ_j are differentiable. This is similar to the proof that solution are differentiable with respect to the initial conditions. We will not do it here. We will show however that if we assume the map Ψ to be differentiable, then the stable manifold W^s is tangent to the stable subspace E^s of $y' = By$ which is $y_{j+1} = \cdots y_n = 0$. To see this we not that $\Psi_j(0, 0) = 0$ and

$$|\psi_j(0, a)| \;=\; \left|\int_0^\infty V(-s)h(u(s, a))\,ds\right| \leq 2\epsilon K^2\|a\|/\sigma\,. \tag{3.70}$$

Since ϵ can be made arbitrarily small by choosing $\|a\|$ small enough, this shows that $\frac{\partial\psi_j}{\partial y_i}(0) = 0$. And so W^s is tangent to E^s.

Example 3.2.4 The proof of the stable manifold theorem provides an algorithm to construct the stable and unstable manifold. Let us consider the system

$$\begin{aligned} x_1' &= -x_1 - x_2^2\,, \\ x_2' &= x_2 + x_1^2\,. \end{aligned} \tag{3.71}$$

Here $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is already in Jordan normal form. So we have

$$U(t) \;=\; \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\,, \quad V(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & e^{t} \end{pmatrix}\,. \tag{3.72}$$

and we can take $a = (a_1, 0)^T$. The integral equation for $u(t, a)$ is given by

$$u(t, a) \;=\; \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \int_0^t \begin{pmatrix} e^{-(t-s)}u_2^2(s) \\ 0 \end{pmatrix} ds \;-\; \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}u_1(s) \end{pmatrix} ds\,. \tag{3.73}$$

Therefore we find

$$\begin{aligned} u^{(0)}(t, a) &= 0\,, \\ u^{(1)}(t, a) &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix}\,, \end{aligned}$$

$$u^{(2)}(t,a) \;=\; \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}e^{-2s}a_1^2 \end{pmatrix} ds \;=\; \begin{pmatrix} e^{-t}a_1 \\ -\frac{1}{3}e^{-2t}a_1^2 \end{pmatrix}.$$

$$u^{(3)}(t,a) \;=\; \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \frac{1}{9}\int_0^t \begin{pmatrix} e^{-(t-s)}e^{-4s}a_1^4 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}e^{-2s}a_1^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}a_1 + \frac{1}{27}(e^{-4t} - e^{-t})a_1^4 \\ -\frac{1}{3}e^{-2t}a_1^2 \end{pmatrix}. \tag{3.74}$$

and one sees that the next term will be $O(a_1^5)$. The stable manifold is given by $\psi_2(a_1) = u_2(0,(a_1,0))$ and is given by

$$\psi_2(a_1) \;=\; -\frac{1}{3}a_1^2 + O(a_1^5). \tag{3.75}$$

Hence the stable manifold is given

$$W^s \;:\; x_2 = -\frac{1}{3}x_1^2 + 0(x_1^5). \tag{3.76}$$

The unstable manifold is obtained by changing t into $-t$ and exchanging x_1 and x_2 and one obtains

$$W^u \;:\; x_1 = -\frac{2}{3}x_1^2 + 0(x_1^5). \tag{3.77}$$

In figure 3.2 we show the approximate manifolds as well as the exact manifolds. Note that the stable and unstable close into a loop (homoclinic loop).

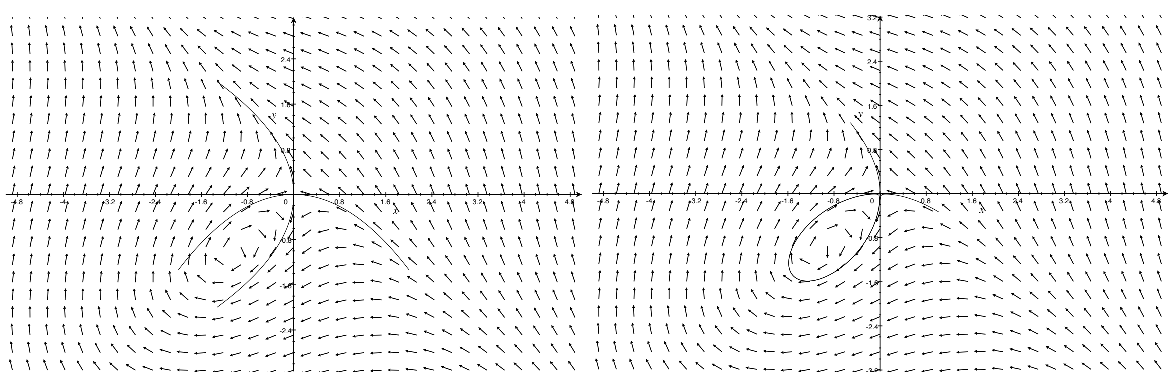

Figure 3.2: The stable and unstable manifolds for the ODE (3.71): on the left the approximate solutions and on the right the global stable and unstable manifold.

3.3 Center manifolds

It is natural to ask whether an analog of Theorem 3.2.2 holds true for $x' = Ax + f(x)$ when the center subspace E^c for $x' = Ax$ is non trivial, i.e., when A has eigenvalues with zero real part. The answer is yes, although there are important differences. The following theorem is quite a bit harder to prove than Theorem 3.2.2 and we will not give the proof here.

Theorem 3.3.1 *Let $f : U \to \mathbf{R}^n$ be of class $\mathcal{C}^1$ where U is an open neighborhood of the origin and let 0 be a critical point, $f(0) = 0$. Suppose that $\frac{df}{dx}(0)$ has k eigenvalues with negative real parts, j eigenvalues with positive real parts, and $m = n-k-j$ eigenvalues with zero real parts. Then there exist*

1. *a m-dimensional center manifold W^c of class $\mathcal{C}^1$ tangent to the center subspace E^c,*

2. *a k-dimensional center manifold W^s of class $\mathcal{C}^1$ tangent to the stable subspace E^s,*

3. *a m-dimensional center manifold W^u of class $\mathcal{C}^1$ tangent to the unstable subspace E^u.*

Furthermore the manifolds W^c, W^s, W^u are invariant under the flow of the ODE $x' = f(x)$.

Center manifolds are different from stable and unstable manifolds: in general they are not uniquely defined. The following example is typical.

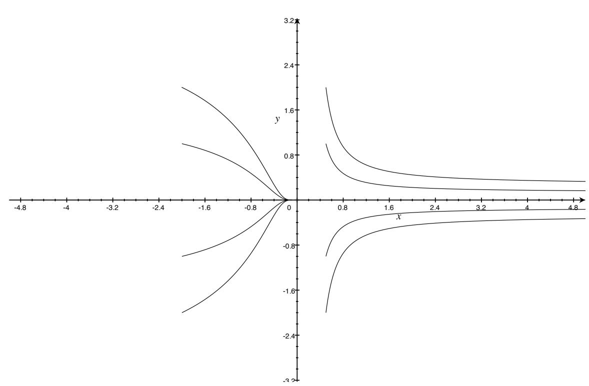

Figure 3.3: The phase portrait for the ODE (3.78).

Example 3.3.2 Consider the system

$$\begin{aligned} x' &= x^2\,, \\ y' &= -y\,. \end{aligned} \tag{3.78}$$

The linearized equation is

$$\begin{aligned} x' &= 0\,, \\ y' &= -y\,. \end{aligned} \tag{3.79}$$

The center subspace is the x-axis, while the stable subspace is the y-axis. The system (3.78) is easily solved: $x(t) = \frac{1}{t-x_0}$, $y(t) = e^{-t}y_0$. Some solution are showed in figure 3.3. Any solution curve of (3.78) with $x_0 < 0$ patched together with the positive x-axis gives a one-dimensional center manifold which is tangent to the center subspace. This simple example shows that the center manifold is *not unique*. There are infinitely many of them.

The stable manifold theorem 3.2.2 gives a a complete description of the dynamics of $x' = f(x)$ in a neighborhood of a hyperbolic critical point. The center manifold theorem does provide such a description; provided we determine the behavior of solutions on the center manifold W^c.

We illustrate how this can be done by considering first the case where E^u is trivial, i.e., $\frac{df}{dx}(0)$ has m eigenvalues with zero real part and k eigenvalues with negative real parts with $m + k = n$. In that case, by a linear change of coordinates, we can assume that the system has the following form

$$\begin{aligned} x' &= Cx + f(x,y)\,, \\ y' &= Px + g(x,y)\,. \end{aligned} \tag{3.80}$$

where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^k$, $f(0,0) = g(0,0) = 0$, $f'(0,0) = g'(0,0) = 0$, the $m \times m$ matrix C has eigenvalues with zero real parts, and the $k \times k$ matrix P has eigenvalues with negative real parts.

The local center manifold is tangent to E^c at 0 and is given by the graph of a function h

$$W^c = \Big\{(x,y) \in \mathbf{R}^m \times \mathbf{R}^k \,:\, y = h(x) \quad \text{for } \|x\| \le \delta\Big\}\,. \tag{3.81}$$

So the flow on the center manifold is given by the set of differential equation

$$x' = Cx + f(x, h(x))\,. \tag{3.82}$$

On the other hand, we can obtain an equation for $h(x)$. Since $y = h(x)$ and W^c is invariant under the flow

$$y' = \frac{dh}{dx}(x)x' \tag{3.83}$$

and substituting the equation for x' and y' gives the nonlinear equation

$$\frac{dh}{dx}(x)\,(Cx + f(x, h(x))) - Ph(x) - G(x, h(x)) \;=\; 0\,. \tag{3.84}$$

The equation for h is first order nonlinear partial differential equation for h. While it is likely to be impossible to solve this equation, one can use it to investigate the qualitative behavior of solutions. The idea is to expand the function $h(x)$ in a power series in x and determine the approximate shape of $h(x)$. We illustrate this with an example

Example 3.3.3 Consider the system

$$\begin{aligned} x' &= x^2y - x^5\,, \\ y' &= -y + x^2\,. \end{aligned} \tag{3.85}$$

In this case $C = 0$ and $P = -1$, $f(x, y) = x^2y - x^5$, $g(x, y) = x^2$. We expand $h(x)$

$$h(x) \;=\; ax^2 + bx^3 + 0(x^4)\,, \frac{dh}{dx}(x) \;=\; 2ax + 3bx^2 + O(x^3)\,, \tag{3.86}$$

and substitute in (3.84)

$$(2ax + 3bx^2 + \cdots)(ax^4 + bx^5 + \cdots - x^5) + ax^2 + bx^3 + \cdots - x^2 \;=\; 0\,. \tag{3.87}$$

Setting the coefficients of like powers to 0 yields $a = 1$, $b = 0$, $c = 0$ and so on. Therefore

$$h(x) \;=\; x^2 + 0(x^5) \tag{3.88}$$

and the equation on the center manifold W^c near the origin is

$$x' \;=\; x^4 + 0(x^5)\,, \tag{3.89}$$

The unstable manifold is easily seen to be the y-axis. From this we easily seen that 0 is unstable, see figure (3.3)

Had we, instead used the center subspace $y = 0$ has an approximation for the center manifold, we would have concluded instead that $x' = -x^5$, from we would have inferred (wrongly) that 0 is stable.

These methods can be easily generalized to the case where E^u is not trivial. By a suitable change of variable we can assume that the ODE has the form

$$\begin{aligned} x' &= Cx + f(x, y)\,, \\ y' &= Px + g(x, y)\,. \\ z' &= Qx + h(x, y)\,. \end{aligned} \tag{3.90}$$

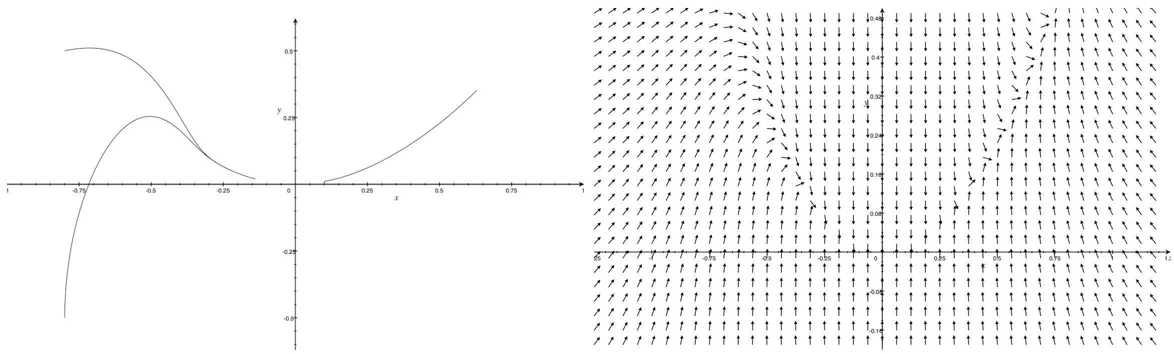

Figure 3.4: some solutions and the he vector field for the ODE (3.85)

where where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^k$, $z \in \mathbf{R}^j$, $f(0,0) = g(0,0) = h(0,0)0$, $f'(0,0) = g'(0,0) = h'(0,0) = 0$, the $m \times m$ matrix C has eigenvalues with zero real parts, the $k \times k$ matrix P has eigenvalues with negative real parts, and the $j \times j$ matrix Q has eigenvalues with positive real parts.

In that case the local center manifold is given by

$$W^c = \left\{(x,y,z) \in \mathbf{R}^m \times \mathbf{R}^k \times \mathbf{R}^j \,:\, y = m_1(x)\,, z = m_2(x) \quad \text{for } \|x\| \leq \delta\right\}. \tag{3.91}$$

where $m_1(0) = m_2(0) = 0$, $\frac{dm_1}{dx}(0) = \frac{dm_2}{dx}(0) = 0$ since W^c is tangent to the center subspace $E^c = \{y = z = 0\}$.

The motion on the center manifold is given by

$$x' = f(x, m_1(x), m_2(x))\,, \tag{3.92}$$

where m_1 and m_2 are solution of the system of partial differential equations

$$\begin{aligned} \frac{dm_1}{dx}(x)\,(Cx + f(x, m_1(x), m_2(x))) - Pm_1(x) - g(x, m_1(x), m_2(x)) &= 0 \\ \frac{dm_2}{dx}(x)\,(Cx + f(x, m_1(x), m_2(x))) - Qm_2(x) - h(x, m_1(x), m_2(x)) &= 0 \end{aligned} \tag{3.93}$$

This can solved to any degree of accuracy by expanding h_1 and h_2 in power series in x.

3.4 Stability by Liapunov functions

The linear stability analysis of the previous section allow to determine asymptotic stability or instability of a critical point by inspection of the linear part of f. There are several questions which cannot be answered by this analysis. If a is an asymptotically stable critical point of linear system, one might want to determine which portion of phase space actually converges to the critical point (i.e., determine its basin of attraction). Another question is to determine the stability of critical point for which some of the eigenvalues of the linearization have a zero real part. In that case the linear stability analysis of the previous section is inconclusive.

Let us consider the autonomous equation

$$x' = f(x)\,, \tag{3.94}$$

where f is locally Lipschitz in an open set U. We assume that $0 \in U$ and $f(0) = 0$. The zero solution $x(t) = 0$ is an equilibrium state. The extension to an equilibrium state $x(t) = a$ is elementary.

We call a function $V : D \to \mathbf{R}$, where D is an open neighborhood of 0, a *Liapunov function* if $V(0) = 0$ and $V(x) > 0$ for $x \in D$, $x \neq 0$.

We recall that the derivative of V along a solution $x(t)$ is given by

$$\frac{d}{dt}V(x(t)) \;=\equiv\; \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x(t))x_i'(t) \;=\; \langle \nabla V(x(t))\,,\, f(x(t))\rangle\,. \tag{3.95}$$

We set

$$LV(x) \;=\; \langle \nabla V(x)\,,\, f(x)\rangle\,. \tag{3.96}$$

We also introduce the concept of *exponential stability* of a solution.

Definition 3.4.1 *A solution $x(t, t_0, x_0)$ is exponentially stable if there exists constants c, $\gamma > 0$ and $\delta > 0$ such that $\|\xi\| \leq \delta$ implies that*

$$\|x(t, t_0, x_0 + \xi) - x(t, t_0, x_0)\| \leq\; ce^{-\gamma(t-t_0)}\,. \tag{3.97}$$

Clearly exponential stability implies asymptotic stability. For linear equation asymptotic stability and exponential stability are equivalent.

Theorem 3.4.2 (Stability Theorem of Liapunov) *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a Liapunov function defined in an open neighborhood D of 0*

1. *If $LV \leq 0$ in D then 0 is a stable critical point.*

2. *If $LV < 0$ in $D \setminus \{0\}$ then 0 is asymptotically stable critical point.*

3. *if $LV \leq -\alpha V$ and $V(x) \geq a\|x\|^{\beta}$ in D (α, β, a are positive constants) then 0 an exponentially stable critical point.*

Proof: 1. Choose η so small that the set $V_\eta \equiv \{x\,;\, V(x) < \eta\}$ is contained in D. Since $V(x) > 0$ for $x \neq 0$ V_η is an open neighborhood of 0. Since $LV \leq 0$ in $V_\eta \subset D$, if $x_0 \in V_\eta$ then the solution $x(t, 0, x_0)$ exists for all $t \geq t_0$ and it does not leave V_η ever. This implies stability since for η small enough we can choose δ and ϵ such that $\{\|x\| < \delta\} \subset V_\eta \subset \{\|x\| < \epsilon\}$.

2. By 1. we know that for η sufficiently small $x(t, t_0, x_0)$ stay in V_η for all times. Moreover $V(x(t, t_0, x_0))$ is a positive decreasing function so that $\lim_{t\to\infty} V(x(t)) = V^*$ exists. Let us assume that $V^* \neq 0$. The set $M = \{V^* \leq V(x) \leq \eta\}$ is a compact set which does not contain 0. We have $x(t) \in M$, for all $t > t_0$ and $\max_{x\in M} LV(x) \leq -\alpha < 0$. So $\frac{dV}{dt}(x(t)) \leq -\alpha$ for all $t > t_0$ which is a contradiction. So we must have $V^* = 0$ and $\lim_{t\to\infty} \|x(t)\| = 0$.

3. We have $\frac{dV}{dt}(x(t)) \leq -\alpha V(x(t))$and thus, by Gronwall Lemma, $V(x(t)) \leq V(x(t_0))e^{-\alpha(t-t_0)}$. Since $b\|x(t)\|^\beta \leq V(x(t))$ we obtain $\|x(t)\| \leq \frac{V(x(t_0))}{b} e^{-\frac{\alpha}{\beta}(t-t_0)}$ and this proves exponential stability. ∎

Theorem 3.4.3 (Instability theorem of Liapunov) *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a function defined in an open neighborhood D of 0 (V is not necessarily positive) which satisfies*

1. $\lim_{\|x\|\to 0} V(x) = 0$.
2. $LV(x) > 0$ *if* $x \in D \setminus \{0\}$.
3. $V(x)$ *takes positive values in each sufficiently small neighborhood of* 0.

Then 0 *is unstable.*

Remark 3.4.4 The conditions 1 and 3 of Theorem 3.4.3 are satisfied, in particular, if V is a Liapunov function, i.e., $V(0) = 0$ and $V(x) > 0$, $x \in D \setminus \{0\}$.

Proof: Stability means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x_0\| \leq \delta$ implies that $\|x(t, 0, x_0)\| \leq \epsilon$ for all $t > 0$.

Let us choose $\epsilon > 0$ such that the set $\{x\,;\, \|x\| \leq \epsilon\} \subset D$. Now for arbitrary $\delta > 0$, by assumptions 1 and 3 we can find x_0 such that $\|x_0\| \leq \delta$ and $V(x_0) = \alpha > 0$. By assumption 2, $LV > 0$, so that $V(x(t)) \geq \alpha$ for all $t \geq 0$. The set $\{x\,;\, \|x\| \leq \epsilon \text{ and } V(x) \geq \alpha\}$ does not contain the origin and therefore there exists $\beta > 0$ such that, in this set, $LV(x) \geq \beta$. We thus obtain that

$$V(x(t)) - V(x(0)) \;=\; \int_0^t LV(x(s))\, ds \geq \beta t\,, \tag{3.98}$$

or

$$V(x(t)) \geq \alpha + \beta t \tag{3.99}$$

as long as $\|x(t)\| \leq \epsilon$. This implies that $x(t)$ actually exits the balls $\{x\,;\, \|x\| \leq \epsilon\}$. Since δ is arbitrary we conclude that 0 is unstable. ∎

Example 3.4.5 Consider the system of equations

$$\begin{aligned} x' &= ax - y + kx(x^2+y^2)\,, \\ y' &= x - ay + ky(x^2+y^2) \end{aligned} \tag{3.100}$$

where $a > 0$ and k are constants. Clearly $(0,0)$ is a critical point and the linearization gives

$$A = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix} \text{ with eigenvalues } \lambda = \pm(a^2-1)\,. \tag{3.101}$$

and so we have

1. If $a^2 > 1$ then 0 is a saddle point and it is unstable by Theorem 3.1.2.
2. If $a^2 = 1$ the system is degenerate.
3. If $a^2 < 1$ then the eigenvalues are purely imaginary and we have a center (vortex).

In order to study the case $a^2 < 1$ we construct a Liapunov function. The linearized system $x' = Ax$ has orbits which are the ellipses

$$V(x,y) = x^2 - 2axy + y^2 = c\,. \tag{3.102}$$

It is quite natural to take $V(x,y)$ a Liapunov function to study the effect of the non-linear terms. We have

$$\begin{aligned} LV &= (2x-2ay)kx(x^2+y^2) + (2y-2ax)ky(x^2+y^2) \\ &= 2k(x^2+y^2)(x^2+y^2-2axy)\,. \end{aligned} \tag{3.103}$$

We conclude from Theorem 3.4.3 that 0 is unstable if $k > 0$ and from Theorem 3.4.2 that 0 is asymptotically stable for $k < 0$.

If a is a asymptotically stable critical point, then all solutions starting in a neighborhood of a converge to a as t goes to infinity. We call a an *attracting point* or an *attractor*. The *basin of attraction* of a is the set of point y such that $x(t,0,y) \to a$ as $t \to \infty$. Liapunov functions are useful to determine, or at least estimate, the basin of attraction of a critical point.

For example, if V is a Liapunov function in a neighborhood D of a and $LV < 0$ in $D \setminus a$, then D is the basin of attraction of a.

Example 3.4.6 Consider the system

$$\begin{aligned} x' &= -x^3\,, \\ y' &= -y(x^2+z^2+1) \\ x' &= -\sin(z) \end{aligned} \tag{3.104}$$

The critical points are $(0, 0, n\pi)$ with $n = 0, \pm 1, \pm 2, \cdots$. Note that if $z = n\pi$ then $z' = 0$ and thus the planes $z = n\pi$ are invariant. Any solution which starts in such plane stays in this plane for all time $t \in \mathbf{R}$. This implies that any solution which starts in the region $|z| < \pi$ stays in this region for ever. Let us study the stability of $0 = (0, 0, 0)$. The linearization around 0 gives the linear system with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.105}$$

and this tells us nothing about the stability of this equilibrium point. However let us consider the function $V(x, y, z) = x^2 + y^2 + z^2$. We have

$$LV = -2x^4 - 2y^2(x^2 + y^2 + z^2) - 2z\sin(z) \tag{3.106}$$

For $|z| < \pi$, $LV < 0$ except at the origin. It follows from Theorem 3.4.2 that the basin of attraction of 0 is the entire region $\{(x, y, z), |z| < \pi\}$.

In many interesting examples, however, $LV = 0$ in some subset of D but nevertheless a is asymptotically stable and to study this we are going to prove a stronger version of the Liapunov stability theorem.

Theorem 3.4.7 (Lasalle Stability Theorem) *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be locally Lipschitz with $f(0) = 0$. Let V be a Liapunov function defined in an open neighborhood D of 0 and let us assume that the set $G = \{x\,; V(x) \leq \alpha, x \in D\}$ is compact for some $\alpha > 0$. Let us assume that $LV(x) \leq 0$ for $x \in G$ and that there is no solution $x(t)$, $t \in \mathbf{R}$ with $x_0 \in G$ on which V is constant. Then 0 is asymptotically stable and V_α is contained in the basin of attraction of 0.*

Proof: Let $x_0 \in G$, then $x(t) = x(t, 0, x_0) \in G$ for all $t > 0$. This follows from the fact that $LV \leq 0$ in G and from the compactness of G (the distance from G to the boundary of D is positive).

The proof is by contradiction. Let us assume that $x(t)$ does not tend to 0. Since $x(t)$ stays in the compact set G there exists a sequence $t_n \to \infty$ so that $\lim_{n\to\infty} x(t_n) = x^*$ with $x^* \in G$.

We claim that the solution $y(t) = x(t, 0, x^*)$ starting at x^* exists for all $t \in \mathbf{R}$ and stays in the set G. Clearly $y(t)$ exists for all positive t. On the other hand $x(t, 0, x(t_n))$ is defined for all $t \in [-t_n, 0]$. Since t_n is an increasing sequence then, for any $k \geq 1$, $x(t, 0, x(t_{n+k}))$ is also defined for all $t \in [-t_n, 0]$. By the continuous dependence on initial conditions we have that $y(t) = x(t, 0, x^*)$ is defined for $t \in [-t_n, 0]$. Since n is arbitrary and $t_n \to \infty$ this proves the claim.

We show next that V is constant on the solution $y(t) = x(t, 0, x^*)$. If $V(x^*) = b$ then $V(x(t_n)) \geq b$ and $\lim_{n\to\infty} V(x_{t_n}) = b$. More generally for any sequence s_n with

$\lim_{n\to\infty} s_n = \infty$ we have $\lim_{n\to\infty} V(x_{s_n}) = b$. This follows from the fact that V is nonincreasing along a solution. With $s_n = s + t_n$ we have $\lim_{n\to\infty} x(s_n) = y(s) = x(s, 0, x^*)$ and therefore $V(x(s, 0, x^*)) = b$. Since s is arbitrary this proves the claim.

This contradicts our assumption that V is not constant on any solution. ∎

Example 3.4.8 Consider the equation $x'' + x' + x^3 = 0$, or $x' = y$ and $y' = -y - x^3$ The equation $x'' + x^3$ is Hamiltonian with Hamiltonian $y^2/2 + x^4/4$. The term $x' = y$ is a friction term and one expects that 0 is an attracting fixed point with basin of attraction $\mathbf{R}^2$. Using the Liapunov function $H(x, y)$ we find that $LH = -y^2$, i.e,. LH is non-positive but vanishes on the line $y = 0$. To apply Lasalle theorem we show that H does not stay constant on any solution, except the trivial solution. By contradiction, assume that $H(x(t), y(t))$ is constant then $LH(x(t), y(t)) = 0$ and thus $y(t) = 0$. The equation $y' = -y - x^3$ implies then that $x(t) = 0$ and this contradicts our assumption that the solution is not trivial. Lasalle Theorem implies that $(0, 0)$ is asymptotically stable and that its basin of attraction is the entire plane $\mathbf{R}^2$.

3.5 Gradient and Hamiltonian systems

There are several classes of systems where the use of Liapunov functions is very natural.

3.5.1 Gradient systems

Let $V : U \to \mathbf{R}$ (U is an open set of $\mathbf{R}^n$) be a function of class $\mathcal{C}^2$. A gradient system on $U \subset \mathbf{R}^n$ is a differential equation of the form

$$x' = -\nabla V(x)\,. \tag{3.107}$$

(The negative sign is a traditional convention). The equilibrium points for (3.107) are the critical points of V, i.e., the points a for which $\nabla V(a) = 0$.

Consider the level sets of the function V, $V^{-1}(c) = \{x\,; V(x) = c\}$. If $x \in V^{-1}(c)$ is a *regular point*, i.e., if $\nabla V(x) \neq 0$, then, by the implicit function Theorem, locally near x, $V^{-1}(c)$ is a smooth hypersurface surface of dimension $n - 1$. For example, if $n = 2$, the level sets are smooth curves.

We summarize the properties of the gradient systems in

Proposition 3.5.1 *Let $V : U \to \mathbf{R}^n$ (U an open set in $\mathbf{R}^n$) be of class $\mathcal{C}^2$ and let $x' = -\nabla V(x)$ be a gradient system.*

1. *If x is a regular point of the level curve $V^{-1}(c)$, then the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$.*

2. *If a is an isolated minimum of V, then it a is an asymptotically stable critical point.*

3. *If a is an isolated minimum of V or a saddle point of V, then a is an unstable critical point.*

Proof: Let y be a vector which is tangent to the level surface $V^{-1}(c)$ at the point x. For any curve $\gamma(t)$ in the level set $V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$ we have

$$0 = \frac{d}{dt}V(\gamma(t))|_{t=0} = \langle \nabla V(x)\,,\, y\rangle\,, \tag{3.108}$$

and so $\nabla V(x)$ is perpendicular to any tangent vector to the level set $V^{-1}(c)$ at all regular points of V. This proves 1.

If $x(t)$ is a solution of (3.107), then we have

$$\frac{d}{dt}V(x(t)) = -\langle \nabla V(x(t))\,,\, \nabla V(x(t))\rangle \leq 0\,. \tag{3.109}$$

If a is an isolated minimum of V, then consider the Liapunov function $W(x) = V(x) - V(a)$. We have $LW(x) < 0$ in a neighborhood of a and so, by Theorem 3.4.2, a is an asymptotically stable equilibrium point. This proves 2. If If a is an isolated minimum of V or a saddle point of V, we consider the function $W(x) = V(a) - V(x)$. If a is a isolated minimum $W(x)$ is a Liapunov function and if a is a saddle point then $W(x)$ satisfy the conditions 1 and 3 of Theorem 3.4.3. In both case we have $LW > 0$ in a neighborhood of a and thus, by Theorem 3.4.3, a is unstable. This proves 3. ∎

Example 3.5.2 Let $V : \mathbf{R}^2 \to \mathbf{R}$ be the function $V(x, y) = x^2(x-1)2 + \frac{y^2}{2}$. The the gradient system is given by

$$\begin{aligned} x' &= -2x(x-1)(2x-1)\,, \\ y' &= -y\,. \end{aligned} \tag{3.110}$$

There are 3 critical points $(0, 0)$, $(0, 1/2, 0)$, and $(0, 1)$. From the form of V one concludes that $(0, 0)$ and $(0, 1)$ are asymptotically stable with basins of attraction $\{-\infty < x < 1/2\,, -\infty < y < \infty\}$ and $\{1/2 < x < \infty\,, -\infty < y < \infty\}$ respectively. The critical point $(1, 0)$ is unstable (saddle point). The solution with initial conditions $x(0) = 1$ and $y(0) = y_0$ satisfy $(x(t), y(t)) \to (0, 1)$. See figure 3.5.2 in the next subsection.

We also have

Proposition 3.5.3 *Let $V : U \to \mathbf{R}^n$ (U an open set in $\mathbf{R}^n$) be of class $\mathcal{C}^2$ and let $x' = -\nabla V(x)$ be a gradient system. If a is a critical point, then the linearization around A has only real eigenvalues.*

Proof: Let a be a critical point. Then the linearized system is given by the matrix $A = (a_i j)$ where

$$a_{ij} \;=\; -\frac{\partial^2}{\partial x_i \partial x_j}(a)\,. \tag{3.111}$$

Since V is of class $\mathcal{C}^2$ we have $a_{ij} = a_{ji}$. Therefore A is a symmetric matrix and its eigenvalues are real. ∎

3.5.2 Hamiltonian systems

Let $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $y = (y_1, \cdots, y_n) \in \mathbf{R}^n$ and let $H \,:\, U \to \mathbf{R}$ (U an an open set of $\mathbf{R}^{2n}$) be a function of class $\mathcal{C}^2$. For mechanical systems x are the coordinates of the particles and y are the momenta of the particles. The function $H(x,y)$ is called the energy or the Hamiltonian of the system.

A Hamiltonian system for the Hamiltonian H is a system of differential equations of the form

$$\begin{aligned} x_i' &= \frac{\partial H}{\partial y_i} \qquad i = 1, \cdots, n\,, \\ y_i' &= -\frac{\partial H}{\partial x_i} \quad\; i = 1, \cdots, n\,. \end{aligned} \tag{3.112}$$

A simple but very important property of Hamiltonian is conservation of energy

Proposition 3.5.4 *Let $H(x,y)$ be a function of class $\mathcal{C}^2$. Let $(x(t), y(t))$ be a solution of the Hamilton equations (3.112), then $H(x(t), y(t))$ is constant.*

Proof: We have

$$\begin{aligned} \frac{d}{dt} H(x(t), y(t)) &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} x_i' + \frac{\partial H}{\partial y_i} y_i' \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial H}{\partial x_i} \;=\; 0\,. \end{aligned} \tag{3.113}$$

Hence H is constant along any solution. ∎

This means that any solution $(x(t), y(t))$ stays on the level set $\{H(x,y) = c = H(x(0), y(0))\}$ of the Hamiltonian. If the level set of the Hamiltonian is compact, then the solution exists for all positive and negative times.

Let $z = (x,y)$ and define $J \,:\, \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ as the linear map given by

$$J \;=\; \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\,. \tag{3.114}$$

where I is the $n \times n$ identity matrix. Note that we have $J^T = J^{-1} = -J$. We can rewrite (3.113) as

$$z' = J^{-1}\nabla H(z)\,. \tag{3.115}$$

Theorem 3.5.5 *Let $U \subset \mathbf{R}^{2n}$ be an open set and $H : U \to \mathbf{R}$ be of class $\mathcal{C}^2$. Then the flow $\phi^t(z) = z(t,0,z)$ for (3.115) satisfy*

$$\left(\frac{\partial\phi^t}{\partial z}\right)^T J \left(\frac{\partial\phi^t}{\partial z}\right) = J\,. \tag{3.116}$$

Remark 3.5.6 A map $T : U \to \mathbf{R}^{2n}$ ($U \subset \mathbf{R}^{2n}$ open) which satisfies $\frac{\partial T}{\partial x}^T J \frac{\partial T}{\partial x} = J$ is called a *symplectic transformation.*

Proof: The derivative $\frac{\partial\phi^t}{\partial z}$ is the solution for the variational equation, which is

$$\Psi'(t) = J^{-1}\frac{d^2H}{dz^2}(z(t,0,z))\Psi(t)\,, \quad \Psi(0) = I\,, \tag{3.117}$$

where

$$\frac{d^2H}{dz^2} = \left(\frac{\partial^2 H}{\partial z_i \partial z_j}\right)_{1\le i,j\le n}\,. \tag{3.118}$$

is a symmetric matrix. We have then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\phi^t}{\partial z}\right)^T J \left(\frac{\partial\phi^t}{\partial z}\right) &= \left(\left(\frac{\partial\phi^t}{\partial z}\right)^T\right)' J \left(\frac{\partial\phi^t}{\partial z}\right) + \left(\frac{\partial\phi^t}{\partial z}\right)^T J \left(\frac{\partial\phi^t}{\partial z}\right)' \\ &= \left(\frac{\partial\phi^t}{\partial z}\right)^T \frac{d^2H}{dz^2}(z(t,0,z))J^2 \left(\frac{\partial\phi^t}{\partial z}\right) + \left(\frac{\partial\phi^t}{\partial z}\right)^T \frac{d^2H}{dz^2}(z(t,0,z)) \left(\frac{\partial\phi^t}{\partial z}\right) \\ &= 0\,. \end{aligned} \tag{3.119}$$

Since (3.116) holds for $t = 0$ it holds thus for all t. ∎

A direct consequence of the symplecticity of the flow is

Theorem 3.5.7 *Let $U \subset \mathbf{R}^{2n}$ be an open set and $H : U \to \mathbf{R}$ be of class $\mathcal{C}^2$. The flow ϕ^t for (3.115) is volume preserving, i.e., for any $A \subset \mathbf{R}^n$ compact with ∂A negligible we have*

$$\mathrm{vol}\left(\phi^t(A)\right) = \mathrm{vol}(A)\,. \tag{3.120}$$

Proof: The map ϕ^t is of class $\mathcal{C}^1$ by Theorem (2.6.6) and injective by the uniqueness of solutions. Since $\det J = 1$, from Theorem 3.5.5 we have

$$\left|\det\left(\frac{\partial \phi^t}{\partial x}\right)\right| = 1\,. \tag{3.121}$$

By the change of variables formula we have

$$\mathrm{vol}\left(\phi^t(A)\right) = \int\int_{\phi^t(A)} dx\,dy = \int\int_{\phi^t(A)} dx\,dy = \int\int_A \left|\det\frac{\partial \phi^t}{\partial x}\right| dx\,dy = \mathrm{vol}(A)\,. \tag{3.122}$$

and so ϕ^t preserves volume. ∎

Remark 3.5.8 This can be derived directly from Liouville Theorem 2.1.6 (see homework). The property of symplecticity is a much stronger than volume preserving, at least for $n \geq 1$.

Remark 3.5.9 Because of the volume preserving property critical points of Hamiltonian systems are never asymptotically stable.

Theorem 3.5.10 *Consider Hamilton's equation (3.112) where H is of class $\mathcal{C}^2$. Let a be a critical point for (3.112). If $H(x,y) - H(a)$ is positive (or negative) in a neighborhood of a then a is a stable critical point.*

Proof: Without loss of generality we may assume that $a = 0$. By Proposition (3.5.4) $V(x,y) = H(x,y) - H(0)$ (or $V(x,y) = H(0) - H(x,y)$) is a Liapunov function with $LV = 0$. The theorem follows immediately from item 1. of Theorem 3.4.2. ∎

In mechanical systems, the Hamiltonian has (usually) the form $H = T + W$ where T is the kinetic energy and U is the potential energy. We have

$$T = \sum_{i=1}^{n} \frac{y_i^2}{2}\,, \quad W = W(x)\,, \tag{3.123}$$

in particular T is positive. The Hamiltonian equations have then the form (Newton's 2^{nd} law)

$$x_i'' = -\nabla W(x)\,. \tag{3.124}$$

Equilibrium solutions then correspond to

$$y = 0\,, \quad \nabla W(x) = 0\,. \tag{3.125}$$

We have

Theorem 3.5.11 *Let $W : U \to \mathbf{R}$ ($U \subset \mathbf{R}^n$ an open set) be a function of class $\mathcal{C}^2$. Assume that a be a critical point of of W, i.e., $\nabla W(a) = 0$. If a is a local (strict) minimum of W then a is a stable critical point for (3.124). If a is a local maximum of W and W is nondegenerate, i.e. if the matrix*

$$\left(\frac{\partial^2 W}{\partial x_i \partial x_j}(a) \right) \tag{3.126}$$

is invertible, then a is an unstable critical point for (3.124).

Proof: Without restricting generality we can assume that $a = 0$ and that $W(0) = 0$. The stability of strict minima follows from Theorem 3.5.10. For local maxima we consider the function

$$V(x, y) = \sum_{i=1}^{n} x_i y_i \tag{3.127}$$

and apply Theorem 3.4.3. It satisfies Condition 1 and 3 of this theorem. The Taylor expansion of W around 0 gives

$$W(x) = \sum_{ij} \frac{\partial^2 W}{\partial x_i \partial x_j}(0) x_i x_j + o(\|x\|^2) . \tag{3.128}$$

and $\left(\frac{\partial^2 W}{\partial x_i \partial x_j}(a) \right)$ is negative definite, i.e., there exists $c > 0$ such that

$$\sum_{ij} \left(\frac{\partial^2 W}{\partial x_i \partial x_j}(0) \right) z_i z_j \le -c \sum_j z_j^2 . \tag{3.129}$$

$$\begin{aligned} LV(x, y) &= \sum_{i=1}^{n} (x_i' y_i + x_i y_i') = \sum_{i=1}^{n} y_i^2 - x_i \frac{\partial W}{\partial x_i} \\ &= \sum_{i=1}^{n} y_i^2 - 2c \sum_{ij} \left(\frac{\partial^2 W}{\partial x_i \partial x_j}(0) \right) x_i x_j + o(\|x\|^2) \\ &\ge \|y^2\| + c\|x^2\| + o(\|x\|^2) , \end{aligned} \tag{3.130}$$

and so $LV > 0$ is positive in a neighborhood of 0. Therefore 0 is unstable.

Example 3.5.12 Let $H : \mathbf{R}^2 \to \mathbf{R}$ be given by $H(p, q) = x^2(x-1)^2 + \frac{y^2}{2}$. The Hamilton's equation of motion are

$$x'' = -2x(x-1)(2x-1) \tag{3.131}$$

There are 3 critical points $(0, 0)$, $(0, 1/2)$, and $(0, 1)$. Both $(0, 0)$ and $(1, 0)$ are local minima of $W(q) = q^2(q-1)^2$ and therefore they are stable. The point $(0, 1/2)$ is a nondegenerate local maximum of $W(q)$ and thus is unstable. In figure 3.5.2 we show the vector field for this Hamiltonian system as well as the vector field for the gradient system with $V = H$. Note that they are perpendicular.

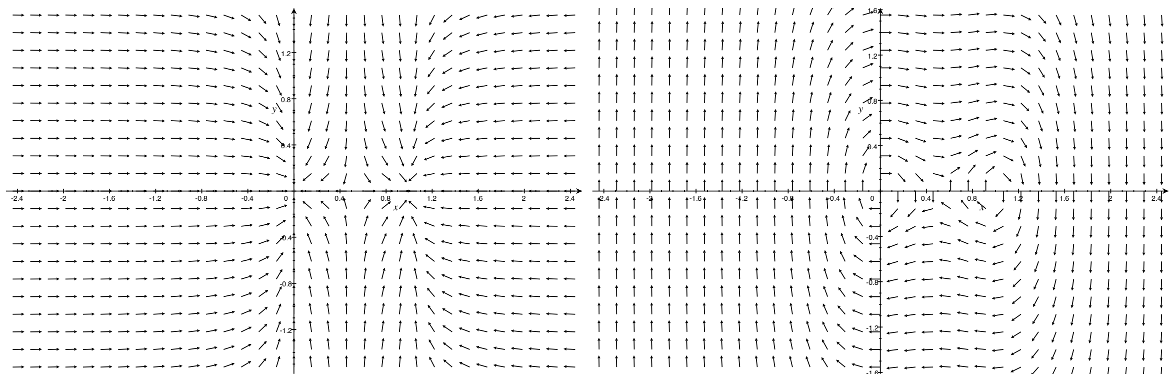

Figure 3.5: The vector field for the gradient system with $V(x,y) = x^2(x-1)^2 + \frac{y^2}{2}$ and the vector field for the Hamiltonian system $H(x,y) = x^2(x-1)^2 + \frac{y^2}{2}$.

Example 3.5.13 Let $H : \mathbf{R}^2 \to \mathbf{R}$ be given by $H(x,y) = 1 - cos(x) + \frac{y^2}{2}$. The Hamiltonian equation are given by

$$x'' = -\sin(x)\,. \tag{3.132}$$

The critical points are $(n\pi, 0)$, $n \in bZ$. For n even $(0, n\pi)$ is a local minimum of $H(x,y)$ and therefore stable. For n odd $n\pi$ is a local maximum of $W(x) = 1 - \cos(x)$, and thus $(0, n\pi)$ is a saddle-point for $H(x,y)$ and thus unstable. The vector field for this Hamiltonian system is shown in figure 3.5.2.

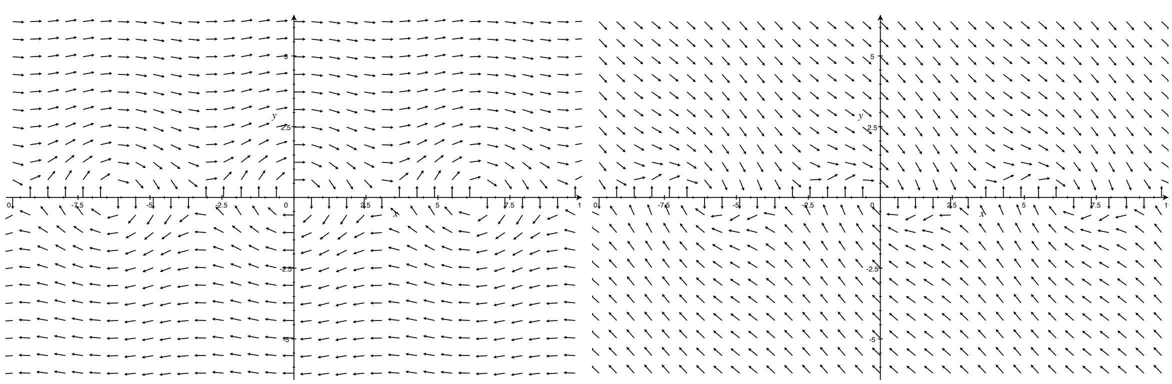

Figure 3.6: The pendulum $x'' + \sin(x) = 0$ and the pendulum with friction $x'' + x' + \sin(x) = 0$.

Example 3.5.14 Let us consider the mathematical pendulum with friction

$$x'' + x' + \sin(x) \,=\, 0\,. \tag{3.133}$$

The corresponding vector field is shown in figure 3.5.2. We could us a linear stability analysis to show that $(0,0)$ is asymptotically stable if $\epsilon > 0$. Instead we use a Liapunov function and estimate, at the same time, the size of its basin of attraction. We consider the Liapunov function $V(x,y) = H(x,y)+1 = y^2/2+1-\cos(x)$. We have $V(0,0)=0$ and $LV \,=\, -y^2 \leq 0$. Since $LV=0$ for $y=0$ the Liapunov function is not strict and we will use Theorem 3.4.7.

Fix a number $c < 2$ and let us consider the set

$$P_c \,=\, \{(x,y)\,|\,V(x,y) \leq c \;\text{ and }\; |x| < \pi\}\,. \tag{3.134}$$

For $c < 2$, the set $\{V(x,y) \leq c\}$ consist of infinitely many disjoints closed regions given by the conditions $\{|x - 2n\pi| < \pi\}$, $n \in \mathbf{Z}$. Thus P_c is compact.

We next show that there is no solution on which V is constant, except the 0 solution. Let us assume that there is such a solution, then we have $\frac{d}{dt}V(x(t),y(t)) \,=\, -y^2(t) \,=\, 0$ and thus $y(t) \,\equiv\, 0$. Thus $x'(t) \,=\, y(t) \,=\, 0$ so $x(t)$ is constant. We also have $y' = -\sin(x) = 0$ and therefore $q(t) \equiv 0$. This is a contradiction. By Theorem 3.4.7 we conclude that $(0,0)$ is asymptotically stable and that P_c is contained in its basin of attraction.

Chapter 4

Poincaré-Bendixson Theorem

We discuss in this chapter the long time limit of two dimensional autonomous systems. In particular, we discuss in the problem of the existence of periodic solutions for two dimensional systems. Closed orbits correspond to periodic solutions and according Jordan closed curve theorem they separate $\mathbf{R}^2$ into two connected components, the interior of and the exterior of the orbits. This makes the 2-dimensional case special and much more tractable then the general case.

4.1 Limit sets and attractors

Let us consider the autonomous equation $x' = f(x)$ where $x \in \mathbf{R}^n$ and $f(x)$ is locally Lipschitz. We denote by $\gamma(x_0)$ the *orbit* corresponding to the solution with $x(0) = x_0$. In particular if $x(t_1) = x_1$ then we have $\gamma(x_0) = \gamma(x_1)$. We denote by $\gamma^+(x_0)$ the *positive orbit* defined by $x(t)$, $t \geq 0$ and by $\gamma^-(x_0)$ the *negative orbit* defined by $x(t)$, $t \geq 0$. We have $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$ and for a periodic solution we have $\gamma^+(x_0) = \gamma^-(x_0)$.

We call a set M invariant, if $x(0) \in M$ implies that $x(t) \in M$ for all $t \in \mathbf{R}$ and we call a set *positively invariant* , if $x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

Definition 4.1.1 A point x^* is called a *positive limit point* of the orbit $\gamma(x_0)$ if there exists an increasing sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that $x^* = \lim_{n\to\infty} x(t_n)$. A *negative limit point* of $\gamma(x_0)$ is defined similarly.

Example 4.1.2 For the equation $x' = -2x$ and $y' = -y$, for all orbits $(0, 0)$ is the unique limit point.

Example 4.1.3 If $x' = f(x)$ has a periodic orbit C, every point on the periodic orbit is a positive and negative limit point of the orbit.

Definition 4.1.4 We denote by $\omega(\gamma)$ the set of all positive limit points for the orbit γ, it is called the *ω-limit set* of γ. Similarly we denote $\alpha(\gamma)$ the set of all negative limit points for the orbit γ (the *α-limit set* of γ).

The basic properties of limit points and limit sets are summarized in

Theorem 4.1.5 *The sets $\omega(\gamma)$ and $\alpha(\gamma)$ are closed and invariant. If the positive orbit γ^+ is bounded, then $\omega(\gamma)$ is a compact, connected and non-empty set. If $\gamma^+ = \gamma^+(x_0)$ then we have $\lim_{t\to\infty} \text{dist}\,(x(t,0,x_0)\,,\,\omega(\gamma)) = 0$. Analogous properties hold for γ^- and $\alpha(\omega)$.*

Proof: We first prove that $\omega(\gamma)$ is closed. Let $\{y_m\}$ be a convergent sequence in $\omega(\gamma)$ with limit y. We show that y is a limit point. For any $\epsilon > 0$, there exists M such that, for any $m \geq M$, $\|y_m - y\| \leq \epsilon/2$ and there exist sequences t_n^m with $\lim_n t_n^{(m)} = \infty$ such that $\|x(t_n^{(m)}) - y_m\| \leq \epsilon/2$. This implies that y is a limit point.

We next show that $\omega(\gamma)$ is an invariant invariant. Let $x^* \in \omega(\gamma)$, then there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\lim_n x(t_n) = x^*$. For arbitrary t, we have $x(t+t_n,0,x_0) = x(t,0,x(t_n,0,x_0))$ so that, using the continuous dependence on initial conditions we have $\lim_n x(t+t_n,x_0) = x(t,0,x^*)$. Therefore for all t, the orbit which contains x^* lies in $\omega(\gamma)$ and this proves invariance.

Let us assume that γ^+ is bounded, then it has at least one accumulation point and so $\omega(\gamma)$ is non empty. Since $\omega(\gamma)$ is closed it is also compact.

Since, for any t, $\lim_n x(t+t_n,0,x_0) = x(t,x^*)$ we have $\gamma(x^*) \subset \omega(\gamma(x_0))$. It follows that $\lim_{t\to\infty} \text{dist}\,(x(t,0,x_0)\,,\omega(\gamma)) = 0$. This also implies that $\omega(\gamma)$ is connected. ∎

Definition 4.1.6 We say that a closed invariant set A is an *attracting set* if there is a open neighborhood U of A such that for all points $x_0 \in U$, $x(t,0,x_0) \in U$ for all $t > 0$ and $\lim_{t\to\infty} \text{dist}\,(x(t)\,,\,A) = 0$. An *attractor* is an attracting set which contains a dense orbit. A *stable limiting cycle* is an attractor which consists of a single periodic orbit.

Example 4.1.7 For a critical point a, we always have $\gamma(a) = a$ and so $\{a\}$ contains a dense orbit. If a is is asymptotically stable then it is an attractor.

Example 4.1.8 Consider the system $x' = x - x^3$ and $y' = -y$. The system has three critical points $(-1,0)$ and $(1,0)$ which are asymptotically stable and $(0,0)$ which is a saddle point. The set $A = \{y = 0\,,\,-1 \leq x \leq 1\}$ is an attracting set, every orbit is attracted to A. But A does not contain a dense orbit and it is not an attractor.

Example 4.1.9 Consider the system

$$\begin{aligned} x' &= -y + x(1-x^2-y^2)\,, \\ y' &= x + y(1-x^2-y^2)\,. \end{aligned} \tag{4.1}$$

It is convenient to write the equation in polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$. A simple computation shows that (4.1) is equivalent to

$$\begin{aligned} r' &= r(1-r^2)\,, \\ \theta' &= 1\,. \end{aligned} \tag{4.2}$$

We see that the origin is an equilibrium point of the system. The set $r = 1$ is invariant and consists of a periodic orbit of period 2π. The origin is unstable and the flow spirals outward (counterclockwise) for $0 < r < 1$ and inward for $r > 1$. The circle $\{r = 1\}$ is a stable limiting cycle. Each point in the open neighborhood $U = \{a < r < A\}$ where $a > 0$ is attracted to the limiting cycle.

Example 4.1.10 Consider the system (in polar coordinates)

$$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= \sin^2(\theta) + (1-r)^3\,. \end{aligned} \tag{4.3}$$

Both the origin and the circle $\{r = 1\}$ are invariant set. The circle $\{r = 1\}$ is the ω-limit set for all orbits starting outside the origin and outside the circle. The invariant set $\{r = 1\}$ consists of four orbits given by $\theta = 0$, $\theta = \pi$, and the arcs $0 < \theta < \pi$, and $\pi < \theta < 2\pi$.

4.2 Poincaré maps and stability of periodic solutions

We begin by considering a periodic system

$$x' = f(t, x)\,, \tag{4.4}$$

where $f(t, x)$ is periodic of period p, $f(t + p, x) = f(t, x)$. Let us assume that f is of class $\mathcal{C}^1$ and that eq. (4.4) has a periodic solution $x_p(t)$ of period p. We linearize around the periodic solution, i.e., we set $y(t) = x(t) - x_p(t)$ so that

$$y'(t) = \frac{df}{dx}(t, p(t))y + h(t, y)\,, \tag{4.5}$$

where

$$h(t, y) = f(t, y + x_p(t)) - f(t, x_p(t)) - \frac{df}{dx}(t, p(t))y\,. \tag{4.6}$$

Note that $h(t, y)$ satisfies $h(t, 0) = 0$ and for any t_0

$$\lim_{\|y\|\to 0} \sup_{t>t_0} \frac{\|h(t, y)\|}{\|y\|} = 0\,. \tag{4.7}$$

The linearized equation is

$$y'(t) = \frac{df}{dx}(t, p(t))y\,, \tag{4.8}$$

where $A(t) = \frac{df}{dx}(t, p(t))$ is periodic of period p. By Floquet theorem, there exists a periodic matrix $P(t)$ and a matrix R such that the transformation $y = P(t)z$ transform the system into the form

$$z' = Az + P^{-1}(t)h(t, P(t)z)\,. \tag{4.9}$$

The stability properties of the 0 solution of (4.9) are thus equivalent to the stability properties of the periodic solution $x_p(t)$. From the stability/instability theorem 3.1.1 and 3.1.2 we conclude that the periodic solution $x_p(t)$ is stable if all eigenvalues of A have negative real parts and that $x_p(t)$ is unstable if at least one eigenvalue of A has positive real part.

This simple and appealing stability analysis however breaks down completely if $x_p(t)$ is a periodic solution of the autonomous system

$$x' = f(x)\,, \tag{4.10}$$

where f is of class $\mathcal{C}^1$. In this case we have

$$y'(t) = \frac{df}{dx}(p(t))y + h(t, y)\,, \tag{4.11}$$

where

$$h(t, y) = f(y + x_p(t)) - f(x_p(t)) - \frac{df}{dx}(p(t))y\,. \tag{4.12}$$

and the linearized equation is

$$y'(t) = \frac{df}{dx}(p(t))y\,. \tag{4.13}$$

As we have already noted in Section 2.5, $x_p'(t)$ is always a periodic solution of the variational equation. This implies that R always has an eigenvalue 0 and therefore $x_p(t)$ cannot be asymptotically stable. This can be also seen as follows: since f does not depend on x, if $x_p(t)$ is a periodic solution, then $y_p(t) = x_p(t+\delta t)$ is also a periodic solution. For small δt, $y_p(t)$ is near $x_p(t)$ at $t = 0$ but $|y_p(t) - x_p(t)|$ does not tend to 0 as $t \to \infty$.

For periodic solutions a much more natural concept of stability is to require that that if C is the periodic orbit corresponding to $x_p(t)$ there exists $\delta > 0$ such that if $\|x_p(0) - x(0)\| \le \delta$ then

$$\lim_{t\to\infty} \mathrm{dist}(x(t), C) = 0\,. \tag{4.14}$$

If, in addition, there exists a number $\alpha \in [0, p)$ such that $\lim_{t\to\infty} \|x(t) - x_p(t+\alpha)\| = 0$, then $x(t)$ is said to have *asymptotic phase* α. This means that the solution $x(t)$ winds around as it approaches the periodic solution $x_p(t)$.

We also introduce the concept of the Poincaré map. Let C be a periodic orbit of the autonomous system

$$x' = f(x)\,, \tag{4.15}$$

corresponding to the a periodic solution $x_p(t)$. Let us assume that the orbit passes passes through the point x_0. Let Ξ be an hyperplane perpendicular to the orbit C at x_0. If $x \in \Xi$ is sufficiently close to x_0, then the solution starting at x at $t = 0$ will cross the hyperplane Ξ at a point $P(x)$ near x_0. The mapping $x \mapsto P(x)$ is called a *Poincaré map*.

There is nothing special about hyperplanes and so the Poincaré map can be defined in a similar way if Ξ is a smooth hypersurface through x_0 which is not tangent to the periodic orbit C.

Theorem 4.2.1 *Let $f : U \to \mathbf{R}^n$ (U an open set of $\mathbf{R}^n$) be of class $\mathcal{C}^1$ and let $\phi^t(x)$ denote the flow defined by the differential equation $x' = f(x)$. Assume that $x_t(p) = \phi^t(x_0)$ is a periodic solution of period p and the orbit $\{\phi^t(x_0)\}_{0\le t\le p}$ is contained in U. Let Ξ be the hyperplane orthogonal to C at x_0, i.e.*

$$\Xi = \{x \in \mathbf{R}^n\,,\, \langle (x - x_0)\,,\, f(x_0)\rangle = 0\}\,. \tag{4.16}$$

Then there exists $\delta > 0$ and a unique function $\tau(x)$ which is defined and continuously differentiable in $N_\delta(x_0) = \{x \in \Xi\,;\, \|x - x_0\| < \delta\}$ so that $\tau(x_0) = p$ and

$$\phi^{\tau(x)}(x) \in \Xi\,, \qquad \text{for all} \qquad x \in N_\delta(x_0)\,. \tag{4.17}$$

Proof: This is an consequence of the smooth dependence of $\phi^t(x)$ with respect to x and t (see Theorem 2.6.6) and from the implicit function theorem. We define the function

$$F(t,x) = \left\langle (\phi^t(x) - x_0)\,,\, f(x_0)\right\rangle\,. \tag{4.18}$$

The function $F(t,x)$ is of class $\mathcal{C}^1$ for $(t,x) \in \mathbf{R} \times U$. Since $\phi^p(x_0) = x_0$ we have

$$F(p,x_0) = 0\,. \tag{4.19}$$

Since $\frac{\partial \phi^t(x_0)}{\partial t}|_{t=p} = f(x_0)$ we have

$$\frac{\partial F}{\partial t}(p,x_0) = \left\langle \frac{\partial \phi^t(x_0)}{\partial t}|_{t=p}\,,\, f(x_0)\right\rangle = \langle f(x_0)\,,\, f(x_0)\rangle \neq 0\,, \tag{4.20}$$

since x_0 is not a critical point. From the implicit function theorem there exists a function $\tau(x)$ of class $\mathcal{C}^1$ defined in a neighborhood $B_\delta(x_0)$ such that $\tau(x_0) = x_0$ and

$$F(\tau(x),x) = \left\langle (\phi^{\tau(x)}(x) - x_0)\,,\, f(x_0)\right\rangle = 0\,, \tag{4.21}$$

i.e.,

$$\phi^{\tau(x)}(x) \in \Xi \,. \tag{4.22}$$

This concludes the proof by taking $N_\delta(x_0) = B_\delta(x_0) \cap \Xi$. ∎

Theorem 4.2.1 implies that the *Poincaré map* $P : N_\delta(x_0) \to \Xi$ given by

$$P(x) \;=\; \phi^{\tau(x)}(x)\,, \tag{4.23}$$

is well defined and of class $\mathcal{C}^1$. Fixed points of the Poincaré map $P(x) = x$ correspond to periodic orbits $\phi^t(x)$ for (4.15). One can also show easily that the map $P(x)$ is invertible with an inverse of class $\mathcal{C}^1$ given $P^{-1}(x) \;=\; \phi^{-\tau(x)}(x)$.

Example 4.2.2 For the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2)\,, \\ y' &= x + y(1 - x^2 - y^2)\,. \end{aligned} \tag{4.24}$$

or equivalently, in polar coordinates

$$\begin{aligned} r' &= r(1 - r^2)\,, \\ \theta' &= 1\,. \end{aligned} \tag{4.25}$$

we can compute the Poincaré map explicitly. The solution to equations (4.25) with $r(0) = r_0$ and $\theta(0) = \theta_0$ are

$$\begin{aligned} r(t, r_0) &= \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-2t}\right]^{-\frac{1}{2}}\,, \\ \theta(t, \theta_0) &= t + \theta_0\,. \end{aligned} \tag{4.26}$$

Let Ξ be the ray $\theta = \theta_0$ through the origin, then Ξ is perpendicular to the closed orbit $r = 1$. Any trajectory starting at (r_0, θ_0) at time 0 intersects Ξ at time 2π. Therefore the Poincaré map is given by

$$P(r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{-\frac{1}{2}}\,. \tag{4.27}$$

with $P(1) = 1$ and

$$P'(r_0) = e^{-4\pi} \frac{1}{r_0^3} \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{-\frac{3}{2}} \tag{4.28}$$

so that $P'(1) = e^{-4\pi} < 1$.

For two-dimensional system, Ξ is a line segment intersecting C orthogonally. We can parametrize the line Ξ by the distance s to the intersection point x_0. We can then write the Poincaré map as $P = P(s)$ as a a function of s which is defined in a neighborhood of the origin and satisfy $P(0) = 0$.

The stability of the periodic orbit C is determined by $P'(0)$. To see this let us introduce the *displacement function*

$$d(s) \,=\, P(s) - s\,. \tag{4.29}$$

Then $d(0) = 0$ and $d'(s) = P'(s) - 1$. If $d'(0) \neq 0$ then, by continuity the sign of $d(s)$ will be the same in a neighborhood of 0. Thus if $d'(0) < 0$ (i.e $P'(0) < 1$) therefore the periodic orbit C is a stable limit cycle since the successive intersection of the orbit with Ξ approach 0. Similarly if $d'(0) < 0$ (i.e., $P'(0) < 1$) C is an unstable limit cycle.

Example 4.2.3 For the example 4.2.2 we have seen that $P'(1) < 1$ which means that the circle $r = 1$ is a stable limit cycle.

This analysis can be generalized easily to higher dimension. Suppose P that the Poincaré $P : N_\delta(x_0) \to \Xi$ satisfies $P(x_0) = x_0$ and that the $n-1$ eigenvalues of the derivative $P'(x_0)$ at x_0 have all absolute value less than 1. Since the map P is differentiable, $P'(x)$ is continuous and therefore for x sufficiently close to x_0 all the eigenvalues of $P'(x)$ have also absolute values less than 1. This implies that the successive approximation of the orbits with Ξ approach 0.

It turns out that the eigenvalues of $P'(0)$ are closely related to the Floquet multipliers of the linearization around the periodic orbit $x_p(t)$.

Theorem 4.2.4 *Let $f : U \to \mathbf{R}^n$ ($U \subset \mathbf{R}^n$ open) be of class $\mathcal{C}^1$ and let assume that $x_p(t)$ is periodic orbit for $x' = f(x)$ which is contained in U. For δ sufficiently small let $P(x) : N_\delta(x_0) \to \Xi$ be the Poincaré map for $x_p(t)$. Let $\lambda_1, \cdots, \lambda_{n-1}$ be the $n-1$ eigenvalues of $P'(0)$ then $\lambda_1, \cdots, \lambda_{n-1}, 1$ are the Floquet multipliers for the linearized equation*

$$x' \,=\, A(t)x\,, \qquad A(t) \,=\, \frac{df}{dx}(x_p(t))\,. \tag{4.30}$$

Proof: Without restricting generality, by a change of variable, we can assume that $x_0 = 0$ and that $f(0) = (0, \cdots, 0, 1)^T$. This means that the hyperplane Ξ is the hyperplane $x_n = 0$.

The flow for $x' = f(x)$ is denoted by $\phi^t(x)$ and $H(t,x) \equiv \frac{\partial}{\partial x}\phi^t(x)$ satisfies the variational equation

$$\frac{\partial}{\partial t}H(t,x) \,=\, \frac{df}{dx}(\phi^t(x))H(t,x)\,, \qquad H(0,x) \,=\, I \tag{4.31}$$

In particular for $x = 0$, $H(t, 0)$ is the resolvent for (4.30). Since $x_p(t)$ is a solution of $x' = f(x)$ then $x_p'(t)$ is a solution of (4.30) with initial condition $x_p'(0) = f(0)$. For $t = p$, $x_p'(p) = f(0)$ and thus

$$H(t, 0)f(0) \;=\; f(0)\,, \tag{4.32}$$

i.e., 1 is an eigenvalue of $H(p, 0)$ with eigenvector $f(0)$. By Floquet Theorem $H(p, 0) = C = e^{pR}$ and thus 1 is a Floquet multiplier with eigenvector $f(0) = (0, \cdots, 0, 1)^T$. Therefore the last column of $H(p, 0)$ is the vector $(0, \cdots, 0, 1)^T$.

Recall that the Poincaré map $P(x)$ is defined to be the restriction to Ξ of the map $h(x)$ where H is given by

$$h(x) \;=\; \phi^{\tau(x)}(x) \tag{4.33}$$

and $\tau(x)$ is the first time the solution hits the hyperplane Ξ (see Theorem 4.2.1). We have

$$\begin{aligned} \frac{dh}{dx}(x) \;&=\; \frac{\partial}{\partial t}\phi^{\tau(x)}(x)\frac{d\tau}{dx}(x) + \frac{\partial}{\partial x}\phi^{\tau(x)}(x) \\ &=\; \frac{\partial}{\partial t}\phi^{\tau(x)}(x)\frac{d\tau}{dx}(x) + H(\tau(x), x)\,, \end{aligned} \tag{4.34}$$

and thus for $x = 0$ we obtain

$$\begin{aligned} \frac{dh}{dx}(0) \;&=\; f(0)\frac{d\tau}{dx}(0) + H(p, 0)\,, \\ &=\; \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \frac{\partial\tau}{\partial x_1}(0) & \cdots & \cdots & \frac{\partial\tau}{\partial x_n}(0) \end{pmatrix} + \begin{pmatrix} & 0 \\ \tilde{H}(p, 0) & \vdots \\ & 0 \\ \cdots & 1 \end{pmatrix} \end{aligned} \tag{4.35}$$

By our choice of coordinates the derivative of $P(x)$ consists of the first $n - 1$ rows and columns of $\frac{dh}{dx}$ and therefore

$$\frac{dP}{dx}(0) \;=\; \tilde{H}(p, 0)\,. \tag{4.36}$$

where $\tilde{H}(p, 0)$ is the matrix consisting of the first $n - 1$ rows and columns of $H(p, 0)$. ∎

Remark 4.2.5 It is in general very difficult to compute the Poincaré map explicitly, and as the previous theorem shows, it is also very difficult to compute the derivative of the Poincaré map since it amounts to computing the Floquet multipliers for a system with periodic periodic coefficients. The concept of the Poincaré map is however very useful, it plays for example a central role int the Poincaré-Bendixson theorem on the existence of periodic orbits for planar systems.

4.3 Bendixson criterion

We consider the autonomous system

$$x' = f(x) \tag{4.37}$$

where f is of class $\mathcal{C}^1$ in an open set $U \subset \mathbf{R}^2$. We derive a simple sufficient condition criterion for a two dimensional not to have a periodic orbit.

Theorem 4.3.1 (Bendixson criterion) *Suppose that the open $D \subset \mathbf{R}^2$ is simply connected and f is of class $\mathcal{C}^1$. The equation (4.37) can have a periodic solution only if* $\operatorname{div} f$ *changes sign in D or* $\operatorname{div} f = 0$ *in D.*

Proof: Suppose that we have a closed orbit C in D and let G be the interior of the orbit C. By the divergence theorem (Gauss Theorem) we have,

$$\int\int_G \operatorname{div} f\, dx = \int_C f \cdot ds = \int_C (f_1 dx_2 - f_2 dx_1) = \int_C \left(f_1 \frac{dx_2}{dt} - f_2 \frac{dx_1}{dt} \right) dt = 0 \tag{4.38}$$

where the last equality follows from the fact that the closed curve C can be parametrized by a solution of (4.37). So the integral on the left side vanishes which implies that div f either vanishes or changes sign in D. ∎

Example 4.3.2 The Lienard equation is given $x'' + f(x)x' + g(x) = 0$ where $f(x)$ and $g(x)$ are Lipschitz continuous. The vector fields $h(x, y) = (y, -f(x)y - g(x))^T$ satisfies $\operatorname{div} f(x, y) = -f(x)$. If $f(x)$ is either positive or negative then Theorem (4.3.1) implies that there is no periodic solution.

Example 4.3.3 For the van der Pol equation $x'' + \epsilon(x^2 - 1)x' + x = 0$ we have $\operatorname{div} f(x, y) = -\epsilon(x^2 - 1)$. So a periodic solution, if it exists, must intersect with the lines $x = 1$ or $x = -1$.

4.4 Poincaré-Bendixson Theorem

In this section we prove

Theorem 4.4.1 *Let $f : U \to \mathbf{R}^2$ (U an open set of $\mathbf{R}^2$) be of class $\mathcal{C}^1$. Assume that the positive orbit γ^+ for the system*

$$x' = f(x), \tag{4.39}$$

is bounded and that the limit set $\omega(\gamma^+)$ does not contain critical points. Then $\omega(\gamma^+)$ is a periodic orbit.

We will split the proof of the theorem in a sequence of lemmas. We first introduce the concept of a transversal line and regular points for (4.39).

Definition 4.4.2 A finite closed segment of a straight line l contained in U is *transversal* for (4.39) if l does not contain any critical points of (4.39) and if the vector field f is not tangent to l at any point of l.

Definition 4.4.3 A point $x \in U$ is *regular* for (4.39) if it is not a critical point for (4.39).

Lemma 4.4.4 *Let x^* be an interior point of a transversal l, then for any $\epsilon > 0$, there exists $\delta > 0$ such that every trajectory passing through a point x with $\|x - x^*\| \le \delta$ crosses l at some time t with $|t| \le \epsilon$.*

Proof: This follows from the continuity of the vector field f. Since l contains only regular points, the vector field on l always points on the same side of l. A sufficiently small neighborhood of l contains also only regular points. This implies that an orbit starting close enough from l will actually crosses it for some positive or negative time. The lemma follows then by the continuous dependence of the solution from initial conditions. ∎

Lemma 4.4.5

1. *If a finite closed arc $\{x(t) \,:\, a \le t \le b\}$ of a trajectory γ intersects a transversal l, it does so at a finite number of points.*
2. *If γ is a periodic orbit which intersects l, it does intersect l only once.*
3. *The successive intersections with l form a monotonic sequence (with respect to the order on the line l).*

Proof: Let us assume that for $t \in [a, b]$, $x(t)$ intersects the transversal at infinitely many points $x_n = x(t_n)$. Then the sequence t_n will have an accumulation point $t^* \in [a, b]$. Passing to a subsequence, also denoted by $\{t_n\}$ we have that $x(t_n)$ converges to $x(t^*)$ in l. But on the other hand we have

$$\lim_{n\to\infty} \frac{x(t_n) - x(t^*)}{t_n - t^*} = x'(t^*) = f(x(t^*)) . \tag{4.40}$$

which is a vector tangent to l at $x(t^*)$. Since $x(t_n), x(t^*) \in l$ this is a contradiction to the fact that l is transversal. This proves 1 and thus a fine segment of an orbit meets l finitely many times.

Let now $x_1 = x(t_1)$ and $x_2 = x(t_2)$ with $t_1 < t_2$ be two successive point of intersection of the orbit with l. Suppose that x_1 is distinct from x_2. Then the arc $\{x(t) \,:\, t_1 \le t \le t_2\}$ together with the closed segment $\overline{x_1 x_2}$ on l forms a closed Jordan curve J. By Jordan closed curve Theorem J separates the plane into two regions, the interior of J and the exterior of J. The points $x(t)$ with $t < t_1$ (and t close to t_1) and the points $x(t)$ with $t > t_2$ (and t close to t_2) will be on the opposite sides of the curve J. It, in fact, remains so for all times $t < t_1$ and $t > t_2$. Suppose for example $x(t)$, $t > t_2$ is inside J. The orbit cannot cross J on the transversal because the flow points inward on the transversal, it cannot cross either on the orbit part of J by uniqueness of solutions. Therefore $x(t)$ must remains inside J for all $t > t_2$. A similar argument holds if $x(t)$, $t < t_1$ is inside J.

From this 2 and 3 follows immediately. ∎

This lemma gives some insight in the structure of possible ω-limit sets in $\mathbf{R}^2$.

Lemma 4.4.6 *If an orbit γ and its ω-limit set $\omega(\gamma)$ have a point in common, then γ is either a critical point or a periodic orbit.*

Proof: Let $x_1 = x(t_1) \in \gamma \cap \omega(\gamma)$. If x_1 is a critical point then $x(t) = x_1$ for all $t \in \mathbf{R}$. If x_1 is a regular point, then we can find a transversal l such that x_1 is an interior point of l. Since $x_1 \in \omega(\gamma)$, there exists a point $x^* = x(t^*)$ with, say, $t^* > t_1 + 2$ which is at distance no more than δ from x_1. By lemma (4.4.4) the orbits passing through x^* will intersect l at a time $t_2 > t_1 + 1$. If $x_2 = x_1$ then $\omega(\gamma)$ is periodic orbit. If $x_2 \neq x_1$, then the piece of orbit $\{x(t)\,;\, t_1 \leq t \leq t_2\}$ intersect l only at a finite number of points, by lemma 4.4.5, item 1. By Lemma 1, item 3, the successive intersections of γ with l are a monotone sequence which tend away from x_1. This contradicts our assumption that $x_1 \in \omega(\gamma)$. ∎

Lemma 4.4.7 *If the ω-limit set $\omega(\gamma)$ of an orbit γ intersect a transversal l, it does so in one point only. If x^* is such an intersection point, we have $\omega(\gamma) = \gamma$ with γ a periodic orbit or there exists a sequence t_n with $\lim_n t_n = \infty$ and $x(t_n)$ tends to x^* monotonically on l.*

Proof: Suppose that $\omega(\gamma)$ intersects l in x^*. If the orbits γ passes through x^* then γ and $\gamma(\omega)$ have x^* in common which is regular point. Thus, by lemma 4.4.6 $\omega(\gamma)$ is a periodic orbit. If not, x^* being in the ω-limit set, there exists a sequence $\{t'_n\}$ with $\lim_n t'_n = \infty$ such that $\lim_n x(t'_n) = x^*$. Then, by Lemma 4.4.4, there exists a sequence t_n with $\lim_n t_n = \infty$ such that $x(t_n) \in l$ and $\lim_n x(t_n) = x^*$ monotonically in l by Lemma 4.4.5.

Now suppose that there exists another intersection point y^* of $\omega(\gamma)$ with l. By the same argument we can construct a sequence $x(s_n)$ which tends monotonically in l to y^*. But in that case we can construct a sequence of intersection of the orbit with l which is not monotone. This contradicts Lemma 4.4.5. ∎

Lemma 4.4.8 *If $\omega(\gamma)$ contains no critical point and $\omega(\gamma)$ contains a periodic orbit C, then $\omega(\gamma) = C$.*

Proof: Let us suppose $\omega(\gamma) \setminus C$ is not empty. Since $\omega(\gamma)$ is connected, C must contain a limit point x^* of the set $\omega(\gamma) \setminus C$. Let l be a transversal which contains x^*, from lemma 4.4.7 it follows that that $\omega(\gamma)$ intersects l only at x^*. Since x^* is a limit point of $\omega(\gamma) \setminus C$, then there exist a point y of $\omega(\gamma) \setminus C$ in arbitrarily small neighborhoods of x^*. By Lemma 4.4.4 an orbit through y will intersect l, since $y \in \omega(\gamma)$ and $\omega(\gamma)$ is an invariant set, an orbit through y belongs also to $\omega(\gamma)$ and this contradicts lemma 4.4.7. ∎

With these preparations we can now conclude the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1: Since γ^+ is bounded, $\omega(\gamma^+)$ is compact, connected and not empty. If γ^+ is a periodic orbit then, by lemma 4.4.8, $\gamma^+ = \omega(\gamma^+)$. Suppose $\gamma^+ \neq \omega(\gamma^+)$. There exists an orbit $C \subset \omega(\gamma^+)$ and the orbit C is contained in compact bounded set F. Therefore the orbit C has itself a limit point x^* which is in $\omega(\gamma^+)$ since $\omega(\gamma^+)$ is closed. Let l be a transversal through x^*, then by Lemma 4.4.7, l intersects $\omega(\gamma^+)$ only at x^*. Since x^* is a limit point of C, by Lemma (4.4.4), l must also intersects C at some point which must be then x^*. This implies that C and $\omega(C)$ have a point in common and so, by lemma 4.4.6, C must be a periodic orbit. By lemma 4.4.8 this implies that $\omega(\gamma^+) = C$. ∎

We can derive several consequences from Theorem (4.4.1). If $\omega(\gamma)$ is a periodic orbit C, we call it a *ω-limit cycle*. We have seen that $\text{dist}(x(t), \omega(\gamma)) \to 0$ as $t \to \infty$, which means that the orbit spirals toward C.

Limit cycles possess a kind of (at least one-sided) stability.

Corollary 4.4.9 *Let C be an ω-limit cycle. If $C = \omega(\gamma(x))$, then there exists a neighborhood O of x such that $C = \omega(\gamma(y))$ for all $y \in O$. The set*

$$\{y \,|\, \omega(\gamma(y)) = C\} \setminus \gamma \tag{4.41}$$

is an open set.

Proof: Let l be a transversal to C. Then there exists a line segment f in l, which is disjoint from C and is bounded by $x(t_1)$ and $x(t_2)$ with $t_1 < t_2$ and such that $x(t)$ does not meet this segment for $t_1 < t < t_2$. Consider now the region A bounded one side by C and on the other side by the closed curve consisting of the orbit segment $\{x(t) t_1 \leq t \leq t_2\}$ and the line segment f. This region is positively invariant and if $y \in A \setminus \gamma$ the orbit passing through y spirals toward C. ∎

If there exists orbits attracted to C starting on both sides of C, then C is an attractor.

Corollary 4.4.10 *Let K be a compact set which is positively invariant, then K contains either a limit cycle or a critical point.*

Proof: If K is positively invariant, then any orbit starting in K is bounded. The corollary follows then from Theorem 4.4.1. ∎

Corollary 4.4.11 *Let C be a periodic orbit and let O be the open region in the interior of C. Then O contains either an equilibrium point or a limit cycle.*

Proof: The $D = O \cup C$ is positively and negatively invariant. If O contains no limit cycle or critical point, then for all $x \in U$ we must have by Poincaré-Bendixson Theorem

$$\omega(\gamma(x)) = \alpha(\gamma(x)) \, = \, C \tag{4.42}$$

so the orbit spirals toward C both for positive and negative times. This is a contradiction. ∎

One can in fact show that 0 always contain a critical point.

4.5 Examples

The theorem of Poincaré-Bendixson implies that the existence of a positively invariant set which does not contain any critical point, must contain some periodic orbit (limit cycles). We consider three examples here.

Example 4.5.1 The first example we consider is

$$\begin{aligned} x' &= x - y - x^3 \,, \\ y' &= x + y - y^3 \,. \end{aligned} \tag{4.43}$$

Let us consider the function $V = \frac{x^2+y^2}{2}$. We have

$$LV \, = \, x^2 + y^2 - x^4 - y^4 \,. \tag{4.44}$$

If $x^2 + y^2 < 1$ then $x^2 < 1$ and $y^2 < 1$ and therefore $x^4 < x^2$ and $y^4 < y^2$. This implies that for $x^2 + y^2 < 1$ we have $LV > 0$.

On the other hand we have $x^4 + y^4 = (x^2 + y^2 - 2x^2y^2)$ and thus

$$LV \, = \, x^2+y^2+2x^2y^2-(x^2+y^2) \, \leq \, 2(x^2+y^2)-(x^2+y^2)^2 = (x^2+y^2)(2-(x^2+y^2)) \,.$$

Thus if $x^2 + y^2 > 2$ we have $LV < 0$.

This implies that the annular region $A = \{1 \leq x^2 + y^2 \leq 2\}$ is positively invariant. Any orbit which starts in this region stays in this region forever. It is not difficult to see that the only critical point is the origin and therefore the annular region does not contain a critical point. The Poincaré-Bendixson theorem implies the existence of a periodic orbit contained in A, see figure 4.1

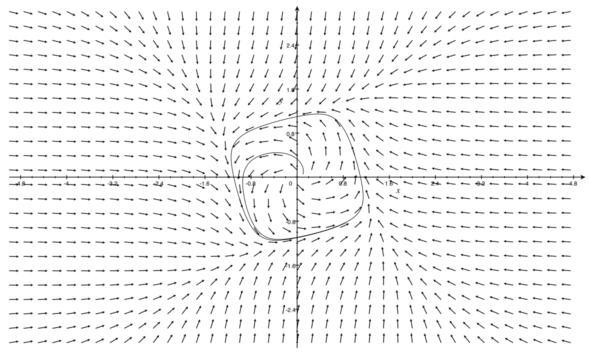

Figure 4.1: The vector field and a solution for the equations (4.43)

We will consider a class of examples, the so-called Lienard equations, for which we establish the existence of periodic orbits. More can be said on such systems with a finer analysis, for example one can determine the number of periodic orbits. The Lienard equation has the general form

$$x'' + f(x)y + g(x)\,. \tag{4.45}$$

where $f(x)$ and $g(x)$ are Lipschitz continuous.

Example 4.5.2 We consider van der Pol type equations

$$x'' + (x^2-1)x' + x^{2n-1}\,, \tag{4.46}$$

where $n \ge 1$, i.e., $f(x) = x^2 - 1$ and $g(x) = x^{2n-1}$. It will be useful to introduce the functions

$$F(x) \;=\; \int_0^x f(z)\,dz \;=\; \frac{x^3}{3} - x\,, \quad G(x) \;=\; \int_0^x g(x) \;=\; \frac{x^{2n}}{2n}\,. \tag{4.47}$$

We will show that a suitable "annular region" around 0 is positively invariant. We will do this using two suitably chosen Liapunov function.

We first consider the function

$$V(x,y) \;=\; \frac{(y+F(x))^2}{2} + G(x)\,, \tag{4.48}$$

We have

$$\begin{aligned} LV(x,y) \;&=\; (y+F(x))\,(-f(x)y - g(x)) + ((y+F(x))f(x) + g(x))\,y \\ &=\; -g(x)F(x) \;=\; -x^{2n-1}(x^3-x)\,. \end{aligned} \tag{4.49}$$

The function $F(x) = x^3 - x$ is negative for $0 < x < 1$ and positive for $-1 < x < 0$ and thus $LV > 0$ if $|x| < 1$ and $x \neq 0$. We choose a so small that the set $\{V(x,y) \le$

$a\}$ is contained in the region $\{|x| \le 1\}$. Then the orbits starting on the boundary $\{V(x,y) \le a\}$ cannot enter this domain.

Next we consider the function

$$W(x,y) \;=\; \frac{y^2}{2} + (F(x) - \arctan(x))\, y + G(x) + \int_0^x f(z)(F(z) - \arctan(z))\, dz\,. \quad (4.50)$$

It is not difficult to see that $\lim_{\|(x,y)\|\to\infty} W(x,y) = \infty$ (look for example along the lines $y = \alpha x$). We have

$$\begin{aligned} LW(x,y) \;&=\; (y + F(x) - \arctan(x))\,(-f(x)y - g(x)) \\ &\quad + \left(\left(f(x) - \frac{1}{1+x^2}\right) y + g(x) + f(x)(F(x) - \arctan(x))\right) y \\ &=\; -(F(x) - \arctan(x))g(x) - \frac{1}{1+x^2} y^2\,. \end{aligned} \quad (4.51)$$

We note that

$$\lim_{\|(x,y)\|_2\to\infty} LW(x,y) \;=\; -\infty\,. \quad (4.52)$$

This implies that there exists b_0 such that on the level sets of $\{W(x,y) = b\}$ for $b \ge b_0$, the vector field points inward the level set. This implies that the system is dissipative, all orbits starting outside the level sets $\{W(x,y) = b\}$ eventually enter it and never leave it again.

This implies that the annular shaped domain

$$A \;=\; \{x\,;\, V(x,y) \ge a\,,\, W(x,y) \le b\} \quad (4.53)$$

is positively invariant. The only critical point is $(0,0)$ so that the Poincaré-Bendixson theorem implies the existence of (at least) one periodic orbit in A. We have also proved that every orbit starting at any point except $(0,0)$ will eventually enter the set A, see figure 4.2.

A somewhat finer analysis shows that, for (4.46) there exist only one periodic orbit which is a stable limiting cycle.

To conclude note that the same argument works for more general function f and g. For example if we assume that there exists $a < b < c$ such that

1. All the zeros of g are contained in $(-a, a)$.

2. In $(-b, -a)$ and (a, b) we have $g(x)F(x) < 0$

3. For $x < -c$ and $x > c$ we have $g(x)F(x) > 0$.

4. $\lim_{x\to\infty} G(x) = \infty$.

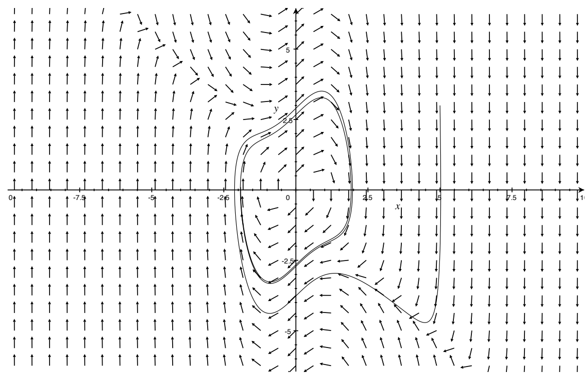

Figure 4.2: The vector field and a solution for the equations (4.46) with $n = 2$

5. For large x, $F(x) = x^{2n+1} +$ lower order terms.

then we have the existence of a periodic orbit. The number of periodic orbits depends on the detailed behavior of F, in particular how many times it changes signs between b and c.

Finally we consider an equation which models chemical reactions involved in the breaking down of sugar in cells.

Example 4.5.3 Let us consider the equations

$$\begin{aligned} x' &= -x + ay + x^2y = f(x,y)\,, \\ y' &= b - ay - x^2y = g(x,y)\,, \end{aligned} \tag{4.54}$$

where a and b are positive constants.

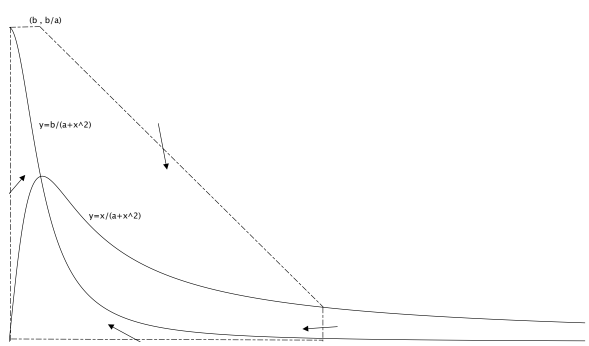

Figure 4.3: The curves $y = x/(a + x^2)$ $y = b/(a + x^2)$ and the region A.

We will be only interested in the region $x \geq 0$, $y \geq 0$. In a first step we will construct a region in the positive quadrant such that the vector field always points inward. The first step is to draw the curves $f(x, y) = 0$ and $g(x, y) = 0$, i.e. the curves $y = x/(a + x^2)$ where $x' = 0$ and $y = b/(a + x^2)$. These curves intersect only at the point $(x, y) = (b, b/a + b^2)$.

Let us consider now the region A bounded by the dotted line in figure 4.3 where the diagonal line has slope -1. On the line segment between $(0, 0)$ and $(0, b/a)$ we have $x' = ay > 0$. On the line segment between $(0, 0)$ and the point C we have $y' = b > 0$. On the line segment between $(0, b/a)$ and $(b, b/a)$ we have $g < 0$ so that $y' < 0$. On the vertical line segment between C until the intersection with the curve $y = x/a + x^2$ we have $f < 0$ so that $x' < 0$. On the diagonal line with slope -1 between $(b, b/a)$ until the intersection with the curve $y = x/a + x^2$ we should compare the relative sizes of x' and y'. Let us consider $x' - (-y')$. We find

$$x' + y' = b - x < 0$$

provided $x > b$. This implies that $dy/dx < -1$ and thus vector field points inward on a line of slope -1.

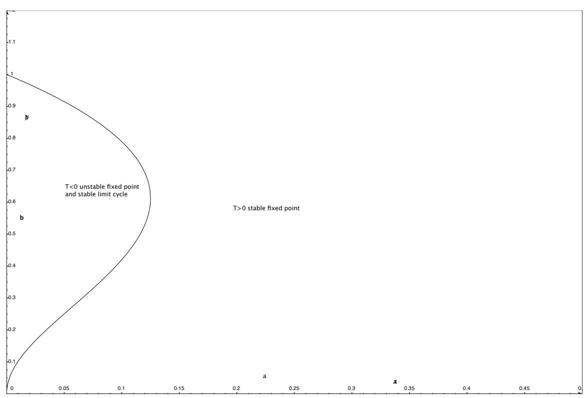

Figure 4.4: The regions in the (a, b) plane with a periodic orbit or a attracting fixed point

Therefore on the boundary of the region A the flow always points inward and thus A positively invariant.

We cannot conclude yet that there is periodic orbit since A contains a critical point. However if the linearization at the critical point has only eigenvalues with positive real parts, there exists a neighborhood of the critical point B such that the vector fields points outward on the boundary of B. In this case the set $A \setminus B$ is positively invariant and we can apply Poincaré-Bendixson and conclude that there exists a periodic orbit.

We have

$$\frac{df}{dx} = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix}. \tag{4.55}$$

and the critical point is $(b, b/(a+b^2))$.

Note that the determinant is $(a+x^2)$ which is always positive. The trace at the critical point is given by

$$\tau = -\frac{b^4+(2a-1)b^2+(a+a^2)}{a+b^2} \tag{4.56}$$

So the fixed point has two eigenvalues with negative real parts if $\tau > 0$. The curve $\tau = 0$, see figure 4.3, is given by

$$b^2 = \frac{1}{2}\left(1-2a \pm \sqrt{1-8a}\right) \tag{4.57}$$

and divide the positive quadrant in the (a, b)-plane into two regions: one with a stable limit cycle and the other with a stable fixed point.

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