Math 645: Midterm

This a take home exam. Closed books, but you can use class notes. Don't talk to each other, but talk to me if you have questions. All the problems are worth an identical number of points.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy a local Lipschitz condition. Show that the equation $x' = f(x)$ has no periodic solution.

2. Let $f : U \to \mathbf{R}^n$ be continuous and satisfy a local Lipschitz condition and let $g : U \to \mathbf{R}$ be positive, continuous and satisfy a local Lipschitz condition. Consider the Cauchy problems

$$\begin{aligned} x' &= f(x)\,, && (1) \\ x' &= g(x)f(x)\,. && (2) \end{aligned}$$

with $x(0) = x_0$. Let $I = I_{\max}$ be the maximal interval of existence containing the origin for the solution $x(t)$ of (1).

   (a) Show that the function $B : I \to \mathbf{R}$ given by

$$B(t) = \int_0^t \frac{1}{g(x(s))}\, ds\,,\,, \tag{3}$$

   is invertible on its range $J \subset \mathbf{R}$.

   (b) Show that if $\rho : J \to I$ is the inverse of B then $y(t) = x(\rho(t))$ is a solution of (2).

   (c) Deduce from (b) that (1) and (2) have the same orbits: i.e. the sets $\{x(t)\}_{t\in I}$ and $\{y(t)\}_{t\in J}$ are identical.

   (d) Suppose $U = \mathbf{R}^n$ and consider the equation $x' = f(x)$. Show that you can always choose $g : \mathbf{R}^n \to \mathbf{R}$ such that the solutions of $x' = g(x)f(x)$ exist for all t.

3. Show that the Cauchy problems for

$$\begin{aligned} x' &= \cos(x)y + 3y - \frac{4xy^2}{1+x^2+y^2}\,, \\ y' &= 3x + \frac{e^{x-y} - e^{x-y}}{e^{x-y} + e^{x-y}}\,, && (4) \end{aligned}$$

and

$$x' = x\left(3 - x^2 - 2y^2\right),$$
$$y' = y\left(1 - x^4 - y^4\right), \tag{5}$$

have unique solution for $t > 0$ for arbitrary initial conditions $x(0) = x_0$, $y(0) = y_0$.

4. Consider the linear inhomogeneous problem

$$x' = Ax + f(t), \tag{6}$$

where $f(t)$ is periodic with period p.

(a) Show that the equation (6) has a unique periodic solution $x_p(t)$ of period p if A has no eigenvalue with real part 0.

(b) Show that if all eigenvalues of A have negative real part the periodic solution found in (a) is asymptotically stable, in the sense that for any solution $x(t)$ of (6) we have

$$\lim_{t\to\infty} \|x(t) - x_p(t)\| = 0.$$

(c) Is there periodic solutions if A has eigenvalues with zero real part?

Hint: Write the solution of (6) using Duhamel's formula.

5. Let A and $B(t)$ be $n \times n$ matrices where A does not depend on t and $B(t)$ is a continuous function of t for $t \in [0, \infty)$.

(a) Show that $x(t)$ is a solution of the Cauchy problem

$$x' = (A + B(t))\,x, \quad x(t_0) = x_0, \tag{7}$$

if and only if $x(t)$ is a solution of the integral equation

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)x(s)\,ds. \tag{8}$$

(b) Suppose that all solutions of the equation $x' = Ax$ are stable and that

$$\int_0^\infty \|B(s)\|\, ds \;=\; K < \infty\,.$$

Show that then all the solutions of $x' = (A + B(t))x$ are stable. *Hint:* Use (a) and Gronwall Lemma (in the version of Homework #2).

(c) Assume that all solutions of the equation $x' = Ax$ are asymptotically stable and that

$$\int_0^\infty \|B(s)\|\, ds \;=\; K < \infty\,.$$

Show that then all the solutions of $x' = (A+B(t))x$ are asymptotically stable. *Hint:* Use (a) and Gronwall Lemma (in the version of Homework #2).

(d) Suppose that all solutions of the equation $x' = Ax$ are stable and that

$$\int_0^\infty \|B(s)\|\, ds \;=\; K < \infty\,.$$

Show the following strengthening of part (b): Let $x(t)$ be a solution $x' = (A+B(t))x$. Then there exists a solution $y(t)$ of $x' = Ax$ such that

$$\lim_{t\to\infty} \|y(t) - x(t)\| \;=\; 0\,.$$

Hint: Without loss of generality you may assume that

$$e^{At} \;=\; U_1(t) + U_2(t)\,,$$

where

$$U_1(t) \;=\; \begin{pmatrix} e^{A_1 t} & 0 \\ 0 & 0 \end{pmatrix}, \qquad U_2(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & e^{A_2 t} \end{pmatrix},$$

and the eigenvalues of A_1 have negative real parts and the eigenvalues of A_2 have zero real parts. Show then that $x(t)$ can be written as

$$x(t) \;=\; e^{A(t-t_0)}v + \int_{t_0}^t U_1(t-s)B(s)x(s)\, ds - \int_t^\infty U_2(t-s)B(s)x(s)\, ds\,.$$