Math 645: Problem set #6

1. Consider the system

$$
\begin{aligned}
x' &= -y + x(1 - x^2 - y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), \\
y' &= x + y(1 - x^2 - y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right).
\end{aligned} \tag{1}
$$

   Show that this system has infinitely many periodic orbits (limit cycles). Determine which ones are stable.

2. Consider the Hamiltonian system with Hamiltonian $H(x, y) = y^2 + W(x)$ and assume that x_0 is an inflection point for $W(x)$. Sketch the corresponding phase diagram in a nieghborhood of $(x_0, 0)$ (it is called a "cusp").

3. Consider the system given in cylindrical coordinates by

$$
\begin{aligned}
r' &= r(1 - r), \\
\theta' &= 1, \\
z' &= -z.
\end{aligned} \tag{2}
$$

   This system has exactly one periodic orbit. Determine it and compute the Poincaré map for the half-plane $y = 0, x > 0$ perpendicular to the periodic orbit. Show that the periodic orbit is asymptotically stable.

4. Show that $(2\cos(2t), \sin(2t))^T$ is periodic orbit for the system

$$
\begin{aligned}
x' &= -4y + x\left(1 - x^2/4 - y^2\right), \\
y' &= x + y\left(1 - x^2/4 - y^2\right),
\end{aligned} \tag{3}
$$

   and show that it is stable.

5. Consider the system given, in polar coordinates by

$$
\begin{aligned}
r' &= r(1 + a\cos(\theta) - r^2), \\
\theta' &= 1.
\end{aligned} \tag{4}
$$

   where $|a| < 1$.

   (a) Show that there exists $0 < r_- < r_+$ such that the annular region $N = \{r_- < r < r_+\}$ is forward invariant.

   (b) Show that the line $S = \{\theta = 0\}$ is a global section and let $P : S \to S$ denote the corresponding Poincaré map.

   (c) Use the Poincaré map and the mean value theorem to show the existence of a periodic orbit. What is the period?

(d) Show that the Floquet multipliers of the periodic orbit are 1 and $e^{-4\pi}$ and thus the orbit is asymptotically stable. *Hint:* To compute $\int_0^{2\pi} r^2 dt$ use the differential equation to set $r^2 = 1 + a\cos\theta - r'/r$.

(e) Conclude that there the periodic orbit found above is actually the only one.

6. Consider the system

$$\begin{aligned} x' &= x - rx - ry + xy\,, \\ y' &= y + rx - ry - x^2\,, \end{aligned} \tag{5}$$

where $r = \sqrt{x^2 + y^2}$. Show that this system can be written in polar coordinates as

$$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= r(1-\cos\theta)\,, \end{aligned} \tag{6}$$

Show that there are two critical points $(0,0)$ (unstable source) and $(1,0)$ (saddle node). Use this information and Poincaré-Bendixson Theorem to show that every solution $x(t)$ which does not pass through the origin satisfy $\lim_{t\to\infty} x(t) = (1,0)$, but that $(1,0)$ is not stable.

7. Consider the system

$$\begin{aligned} x' &= \lambda x - y - xr^2 + \lambda\frac{x^3}{r^3}\,, \\ y' &= X + \lambda y - yr^2 + \lambda\frac{x^2 y}{r^3}\,, \end{aligned} \tag{7}$$

where $r = \sqrt{x^2 + y^2}$. Show that the system has a periodic orbit. *HInt:* Find a invariant annulus region.