Math 645: Problem set #5

1. Consider the ODE in $\mathbf{R}^2$ given, in polar coordinates, by

$$\begin{aligned} r' &= r(1-r)\,, \\ \theta' &= \sin^2(\theta/2)\,. \end{aligned} \tag{1}$$

   Prove that every initial condition except the unstable equilibrium $(x,y)=(0,0)$ is attracted to $(1,0)$ but that $(1,0)$ is not a stable equilibrium.

2. Find all the homoclinic and heteroclinic orbits for the Hamiltonian system with Hamiltonian $H(x,y)=\frac{1}{2}(x^2+y^2)-x^4$. For each equilibrium determine what are the stable and unstable sets W^s and W^u.

3. Consider the FitzHugh-Nagumo equation

$$\begin{aligned} x_1' &= f_1(x_1,x_2) = g(x_1)-x_2\,, \\ x_2' &= f_2(x_1,x_2) = \sigma x_1 - \gamma x_2\,, \end{aligned} \tag{2}$$

   where σ and γ are positive constants and the function g is given by $g(x)=-x(x-1/2)(x-1)$. Show that as the ratio σ/γ decreases the system undergoes a bifurcation from one equilibrium state to three equilibrium states. Compute the critical points and determine their stability properties. Some of the computations are lengthy and you might want to use a geometric argument to determine stability: just look at the directions of the vector field! It is also good idea to make a graph of the orbits before and after the bifurcation (use matlab or mathematica).

4. Consider the system

$$\begin{aligned} x' &= x(1-x-(1+\delta)y-(1-\delta)z)\,, \\ y' &= y(1-y-(1+\delta)z-(1-\delta)x)\,, \\ z' &= z(1-z-(1+\delta)x-(1-\delta)y)\,, \end{aligned} \tag{3}$$

   where $\delta > 0$.

   (a) Determine the equilibria and their stability (there are 8 of them).

   (b) Show that $R=x+y+z$ satisfies a self-contained differential equation and that $R(t)\to 1$ as $t\to\infty$ if $R(0)\neq 0$.

   (c) Show that on the set $R=1$ the system can be reduced to a Hamiltonian system with $H(x,y)=\delta xy(1-x-y)$.

   (d) Discuss completely the dynamics of the system in the positive octant.

5. Consider the equation $x'=Ax+g(x)$.

   (a) Let $\hat{x}(\tau)\equiv x(-\tau)$ and obtain and ODE for $\hat{x}$. Show that stable manifold theorem for this new equation implies an unstable manifold theorem for the original equation.

   (b) Transform back to $t=-\tau$ to obtain an integral equation for the unstable manifold. (Watch for all the minus signs...).

6. Consider the system of equations

$$\begin{aligned} x_1' &= -x_1\,, \\ x_2' &= -x_2 + x_1^2\,, \\ x_3' &= x_3 + x_2^2\,. \end{aligned} \tag{4}$$

   Compute the first four approximations $u^{(j)}(t,a)$ for the functions defining the stable manifold. Show that $u^{(3)}(t,a) = u^{(4)}(t,a)$ and thus $u(t,a) = u^{(3)}(t,a)$. Determine then the local stable and unstable manifolds W^s_{loc} and W^u_{loc}.

7. Use the center manifolds to determine the qualitative behavior near the origin for the equation

$$\begin{aligned} x' &= xy\,, \\ y' &= -y - x^2\,. \end{aligned} \tag{5}$$

8. Construct a Lyapunov function to determine the stability of the equilibrium $(0,0)$ for the following system

$$x' = -x + y - y^2 - x^3\,, \quad y' = x - y + xy \tag{6}$$

9. Consider the equation

$$x'' + x' + 4x^3 - 6x^2 + 2x = 0 \tag{7}$$

   Find the critical points, determine their stability properties and their basin of attraction. *Hint:* Use Lasalle stability theorem.

10. Let $x,\, y \in \mathbf{R}^n$ and consider the Hamiltonian $H(x,y) = \sum_{i=1}^n \frac{y_i^2}{2} + W(x)$ where $W(x)$ is of class $\mathcal{C}^2$. Assume that a is a nondegenerate critical point of W, i.e.

$$\nabla W(a) = 0 \quad \text{and} \quad \det\left(\frac{\partial^2 W}{\partial q_i \partial q_j}(a)\right) \neq 0\,. \tag{8}$$

    and consider the Hamiltonian equation

$$x'' = -\nabla W(x)\,. \tag{9}$$

    Using *linearization* show that $(a,0)$ is an unstable critical point if a is local maximum or a saddle point of W. *Hint:* Study the eigenvalues!