Math 645: Problem set #4

1. (a) Consider the linear inhomogenous equation $x' = Ax + f(t)$ where $f(t)$ is continuous and periodic with period p. Show that the system has a unique solution $x_p(t)$ of period p if A has no eigenvalue which is a multiple $2i\pi/p$.
   Hnt: Use Duhamel's formula to reduce the existence of a solution to the equation $x_0 = e^{pA}x_0 + W$ for the initial condition x_0.

   (b) Consider the second order equation $x'' + bx + kx = g(t)$ with $b \geq 0$ and $k \geq 0$ and $g(t)$ periodic of period p. Determine for which values of b and k the equation has a periodic solution of period p.

   (c) Show that if all the eigenvalues of A have negative real part then every solution $y(t)$ of $x' = Ax + f(t)$ converges to periodic solution found in A, i.e. $\lim_{t\to\infty} y(t) - x_p(t) = 0$.
   Hint: Use Duhamel's formula.

2. Consider the scalar equation (i.e. $n = 1$) $x' = f(t)x$ where $f(t)$ is continuous and periodic of period p.

   (a) Determine $P(t)$ and R in the decomposition of the resolvent given by Floquet Theorem.

   (b) Give necessary and sufficient conditions in terms of $f(t)$ for the solutions to be bounded as $t \to \pm\infty$ or to be periodic.

3. Compute the resolvant $R(t, 0)$ (in real representation) for the ODE

$$\begin{aligned} x' &= \cos(t)x - \sin(t)y\,, \\ y' &= \sin(t)x + \cos(t)y\,. \end{aligned} \tag{1}$$

   and determine $P(t)$ and R in Floquet Theorem *Hint*: Find an equation for $z = x + iy$.

4. Consider the equation $x' = A(t)x$ where A is periodic of period p. Show that a solution $x(t)$ is asymptotically stable if and only if the Floquet multipliers have absolute value less than 1.

5. Consider the differential equation $x'' + \epsilon f(t)x = 0$, where $f(t)$ is periodic of period 2π and

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t < \pi \\ 0 & \text{if} \quad \pi < t \leq 2\pi \end{cases}\,. \tag{2}$$

   For both $\epsilon = 1/4$ and $\epsilon = 4$

   (a) Consider the fundamental solution $\Phi(t)$ which satisfies $\Phi(0) = \mathbf{I}$ and compute the corresponding transition matrix $C = e^{pR}$.

   (b) Compute the Floquet multipliers (the eigenvalues of C).

   (c) Describe the behavior of solution.

6. Let $A(t)$ be periodic of period p and consider ODE $x' = A(t)x$.

   (a) Show that the transition matrix C depends on the fundamental solution, but that the eigenvalues of $C = e^{pR}$ are independent of this choice.

(b) Show that for each Floquet multiplier λ (the eigenvalue of C), there exists a solution of $x' = A(t)x$ such that $x(t+p) = \lambda x(t)$, for all t.

7. Consider the equation $x' = A(t)x$ where $A(t)$ is periodic of period p.

(a) Use Floquet Theorem and Liouville Theorem to show that

$$\det(e^{pR}) = e^{\int_0^p \mathrm{Trace}(A(s))\, ds} . \tag{3}$$

(b) Deduce from (a) that the characteristic exponents μ_i satisfy

$$\mu_1 + \cdots + \mu_n = \frac{1}{p} \int_0^p \mathrm{Trace}(A(s))\, ds \tag{4}$$

8. Show that the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2)\,, & (5)\\ y' &= x + y(1 - x^2 - y^2)\,, & (6)\\ z' &= z\,. & (7) \end{aligned}$$

is expressed in cylinder coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ by

$$\begin{aligned} r' &= r(1 - r^2)\,, & (8)\\ \theta' &= 1, & (9)\\ z' &= z\,. & (10) \end{aligned}$$

It is easy to verify that $\phi(t) = (-\sin(t), \cos(t), 0)$ is a periodic orbit. Determine the corresponding variational equation$\Psi' = A(t)\Psi = f'(\phi(t))\Psi$ and solve it.

9. Consider the equation for the mathematical pendulum

$$x'' + \sin(x) = 0\,, \quad x(0) = \epsilon\,, x'(0) = 0\,, \tag{11}$$

where ϵ is supposed to be small. Show that the solution can be written in the form

$$x(t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + O(\epsilon^4)\,. \tag{12}$$

Compute $x_1(t)$, $x_2(t)$, and $x_3(t)$. *Hint:* Taylor expansion.