Math 645: Homework 3

1. **Continuous dependence on parameters.** Consider the IVP $x' = f(t, x, \mu)$, $x(t_0) = x_0$ where $f : V \to \mathbf{R}^n$ ($V \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ an open set). We denote by $x(t, \mu)$ the solution of the IVP (we have suppressed the dependence on (t_0, x_0)). Let us assume

   - f is a continuous function on V.
   - $f(t, x, c)$ satisfies a local Lipschitz condition in the following sense: Given $(c_0, t_0, x_0) \in V$ and positive constants a, b, c such that $A \equiv \{(t, x, \mu)\,;\, |t - t_0| \le a\,,\, \|x - x_0\| \le b\,,\, \|\mu - \mu_0\| \le c\} \subset V$ then there exists a constant L such that $\|f(t, x, \mu) - f(t, y, \mu)\| \le L\|x - y\|$ for all $(t, x, \mu), (t, y, \mu) \in A$.

   Show that $x(t, \mu)$ depends continuously on μ for t in some interval J containing t_0.

2. If $A \in \mathcal{L}(\mathbf{K}^n)$ the *spectral radius of* A, $\rho(A)$, is defined by

$$\rho(A) \;=\; \max\{|\lambda|\,;\, \lambda \text{ eigenvalue of } A\}\,. \tag{1}$$

   (a) Let

$$A \;=\; \begin{pmatrix} 0.999 & 1000 \\ 0 & 0.999 \end{pmatrix} \tag{2}$$

   Compute the spectral radius of A as well as $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$. Find a norm on $\mathbf{R}^n$ such that $\|A\| \le 1$.

   (b) Show that for any norm on $\mathbf{K}^n$ we have the inequality $\rho(A) \le \|A\|$.

   (c) Show that if A is symmetric ($A^* = A$) then we have the equality $\|A\|_2 = \rho(A)$.

   (d) Show that for any A and any $\epsilon > 0$, there exists a norm such that $\|A\| \le \rho(A) + \epsilon$. *Hint:* You may use (without proof) the fact that there exists a matrix D such that DAD^{-1} is upper triangular (or maybe even in Jordan normal form). Consider the diagonal matrix S with entries $1, \mu^{-1}, \cdots, \mu^{1-n}$. Set $\|x\|_\mu = \|SDx\|$ where $\|\cdot\|$ is any norm on $\mathbf{R}^n$.

3. We have shown in class, using the binomial theorem, that if the matrices A and B commute then $e^{A+B} = e^A e^B$. Here you will show, using a different method based on uniqueness of solutions for ODE that $e^{A+B} = e^A e^B$ if and only if $AB = BA$.

   (a) Let $F(t) = Be^{tA}$ and $G(t) = e^{tA}B$. Show that if A and B commute then $F(t)$ and $G(t)$ satisfies the same ODE and thus must be equal.

   (b) Let $\Phi(t) = e^{tA}e^{tB}$ and $\Psi(t) = e^{t(A+B)}$. Show that if A and B commute, then $\Phi(t)$ and $\Psi(t)$ satisfies the same ODE and thus must be equal.

   (c) Show that if $\Phi(t) = \Psi(t)$ then A and B commute.

4. Show that if $A(t)$ is antisymmetric, i.e., $A^T = -A$, then the resolvant of $x' = A(t)x$ is orthogonal. *Hint:* Show that the scalar product of two solutions is constant.

5. **(D'Alembert reduction method)**. Consider the ODE $x' = A(t)x$ where $A(t)$ is a $n \times n$ matrix and assume that we know one non-trivial solution $x(t)$. Show that one can reduce the equation $x' = A(t)x$ to the problem $z' = B(t)z$ where $z \in \mathbf{R}^{n-1}$ and $B(t)$ is a

$(n-1)\times(n-1)$ matrix. *Hint:* Without loss of generality you may assume that the n^{th} component of $x(t)$, $x_n(t)\neq 0$. Look for solutions of the form $y(t)=\phi(t)x(t)+z(t)$, where $\phi(t)$ is a scalar function and z has the form $z=(z_1,\cdots,z_{n-1},0)^T$.

6. (a) Using the previous problem, compute the resolvent $R(t,1)$ of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x \,, \tag{3}$$

using the fact that $x(t) = (t^2,-t)^T$ is a solution. *Hint:* The solution is

$$\begin{pmatrix} t^2(1-\log t) & -t^2\log t \\ t\log t & t(1+\log t) \end{pmatrix} \tag{4}$$

(b) Compute the solution of

$$x' = \begin{pmatrix} \frac{1}{t} & -1 \\ \frac{1}{t^2} & \frac{2}{t} \end{pmatrix} x + \begin{pmatrix} t \\ -t^2 \end{pmatrix} , \tag{5}$$

with initial condition $x(1)=(0,0)^T$.

7. Compute the resolvant e^{At} for the equations $x'=Ax$ with

(a)

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} . \tag{6}$$

(b)

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} . \tag{7}$$

(c)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} . \tag{8}$$

(d)

$$A = \frac{1}{9}\begin{pmatrix} 14 & 4 & 2 \\ -2 & 20 & 1 \\ -4 & 4 & 20 \end{pmatrix} \tag{9}$$

Hint: All eigenvalues are equal to 2.

8. The equation of motion of two coupled harmonic oscillators is

$$\begin{aligned} x_1'' &= -\alpha x_1 - \kappa(x_1 - x_2) \,, \\ x_2'' &= -\alpha x_2 - \kappa(x_2 - x_1) \,. \end{aligned} \tag{10}$$

This system is a Hamiltonian system. Find the Hamiltonian function. Find a fundamental matrix for this system. You can either write it as a first order system and compute the chararcteristic polynomial or, better, stare at the equation long enough until you make a clever Ansatz. Discuss the solution in the case where $x_1(0)=0$, $x_1'(0)=1$, $x_2(0)=0$, $x_2'(0)=0$.

9. Consider the linear differential equation

$$x' = A(t)x\,, \quad A(t) = S(t)^{-1}BS(t) \tag{11}$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} \tag{12}$$

(a) Show that, for any t, all eigenvalues of $A(t)$ have a negative real part.

(b) Show, that for a suitable choice of a, the differential equation (11) has solutions $x(t)$ which satisfy $\lim_{t\to\infty} \|x(t)\| = \infty$. *Hint*: Set $y(t) = S(t)x(t)$.