Math 645: Problem Set 2

1. Consider the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, where $f(t,x)$ is a continuous function. Show that if the initial value problem has a unique solution then the Euler polygons $x_h(t)$ converge to this solution.

2. Consider the the Cauchy problem $x' = f(t,x)$, $x(0) = 0$ where f is given by

$$f(t,x) = \begin{cases} 0 & \text{if } t \leq 0, \quad x \in \mathbf{R} \\ 2t & \text{if } t > 0, \quad x \leq 0 \\ 2t - \frac{4x}{t} & \text{if } t > 0, \quad 0 \leq x < t^2 \\ -2t & \text{if } t > 0, \quad t^2 \leq x \end{cases} \tag{1}$$

   (a) Show that f is continuous. What does that imply for the Cauchy problem?

   (b) Show that f does not satisfy a Lipschitz condition in any neighborhood of the origin.

   (c) Apply Picard-Lindelöf iteration with $x_0(t) \equiv 0$. Are the accumulation points solutions?

   (d) Show that the Cauchy problem has a unique solution. What is the solution?

   This problem shows that existence and uniqueness of the solution does not imply that the Picard-Lindelöf iteration converges to the unique solution.

3. Consider the Cauchy problem $x' = \lambda x$, $x(0) = 1$, with $\lambda > 0$ and $t \in [0,1]$. Compute the Euler polygons $x_h(t)$ with $h = 1/n$ and show that

$$\frac{\lambda}{1+\lambda h} x_h(t) \leq \frac{dx_h}{dt}(t) \leq \lambda x_h(t)\,. \tag{2}$$

   Deduce from this the classical inequality

$$\left(1 + \frac{\lambda}{n}\right)^n \leq e^\lambda \leq \left(1 + \frac{\lambda}{n}\right)^{n+\lambda} \tag{3}$$

   Hint: Use Gronwall Lemma.

4. Let a, b, c, and d be positive constants. Consider the Predator-Prey equation $x' = x(a-by)$, $y' = y(cx-d)$ with positive initial conditions $x(t_0) > 0$ and $y(t_0) > 0$. Show that the solutions exists for all t and that the solution curves $x(t), y(t)$ are periodic. *Hint:* You can use the change of variables $p = \log(x)$ and $q = \log(y)$

5. (a) Show that any second order ODE $x'' + f(x) = 0$ can be written as a Hamiltonian system for the Hamiltonian function $H(x,y) = y^2/2 + V(x)$, where $y = x'$ and $V(x) = \int_0^x f(t)dt$

   (b) Compute the Hamiltonian function, and it level curves and draw the solutions curves for the following ODE's

      i. $x'' = -\omega^2 x$ (the harmonic oscillator)

ii. $x'' = -a\sin(x)$ (the mathematical pendulum: One end A of weightless rod of length l is attached to a pivot, and a mass m is attached to the other end B. The system moves in a plane under the influence of the gravitational force of amplitude mg which acts vertically downward. Here $x(t)$ is the angle between the vertical and the rod and $a = g/l$).

iii. $mr'' = -\gamma Mm/r^2$ (Vertical motion of a body of mass m in free fall due to the gravity of a body of mass M).

Depending on the energy $H(x_0, y_0)$ of the initial condition discuss in details the different types of solutions which can occur. Are the solutions bounded or unbounded? Are there constant solutions or periodic solutions? Do the solutions converge as $t \to \pm\infty$?

6. (a) Consider the Hamiltonian function $H(x, y) = y^2/2 + V(x)$. Suppose that we have initial conditions $x(0) = x_0$ and $x'(0) = y_0 > 0$ with initial energy $E = H(x_0, y_0)$. Use the conservation of energy to show the solution $x(t)$ is given (implicitly) by the formula

$$t = \int_{x_0}^{x(t)} \frac{1}{\sqrt{2(E - V(s))}}\, ds\,.$$

(b) Assume that $V(x) = V(-x)$, i.e., V is an even function. Show that if $x(t)$ is a solution then so are $x(c-t)$ and $-x(t)$. Furthermore show that if $x(c) = 0$ then $x(c+t) = -x(c-t)$ and that if $x'(d) = 0$ then $x(d+t) = x(d-t)$.

(c) Assume that $V(x) = V(-x)$ and consider periodic solutions. We denote by R the largest swing, i.e., the maximal positive value of $x(t)$ along the periodic solution. Using (a) show that the period p of the periodic solution is given by

$$p = 4\int_0^R \frac{1}{\sqrt{2(V(R) - V(s))}}\, ds$$

Hint: Consider the quarter oscillation starting at the point $x(0) = 0$ and $y(0) = y_0 > 0$ and ending at $x(T) = R > 0$ and $y(T) = 0$. Use also the symmetry of V and (b).

(d) Use (c) to show the period for the harmonic oscillator is independent of the energy E.

(e) Use (c) to show that for the mathematical pendulum the period is given by

$$p = \frac{4}{\sqrt{a}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2(u)}}\, du$$

where $k = \sin r/2$. This integral is an elliptic integral of the first type. *Hint:* Use $1 - \cos(\alpha) = \sin^2\frac{\alpha}{2}$ and the substitution $\sin s/2 = k\sin u$.

7. Show that the following ODE's have global solutions (i.e., defined for all $t > t_0$).

(a) $$\begin{aligned} x' &= 4y^3 + 2x \\ y' &= -4x^3 - 2y - \cos(x) \end{aligned}\,.$$

(b) $x'' + x + x^3 = 0$.

(c) $x'' + x' + x + x^3 = 0$.

(d) $\begin{array}{l} x' = \frac{\sin(2t^2 x)x^3}{1+t^2+x^2+y^2} \\ y' = \frac{x^2 y}{1+x^2+y^2} \end{array}$.

(e) $\begin{array}{l} x' = 5x - 2y - y^2 \\ y' = 2y + 6x + xy - y^3 \end{array}$.

8. Prove the following generalizations of Gronwall Lemma.

- Let $a > 0$ be a positive constant and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

$$g(t) \le a + \int_0^t h(s)g(s)\, ds\,. \tag{4}$$

Then, for any $t \in [0, T]$

$$g(t) \le a e^{\int_0^t h(s)\, ds}\,. \tag{5}$$

- Let $f(t) > 0$ be a positive function and $g(t)$ and $h(t)$ be nonnegative continuous functions. Suppose that for any $t \in [0, T]$

$$g(t) \le f(t) + \int_0^t h(s)g(s)\, ds\,. \tag{6}$$

Then, for any $t \in [0, T]$

$$g(t) \le f(t) e^{\int_0^t h(s)\, ds}\,. \tag{7}$$

9. Consider the FitzHugh-Nagumo equation

$$\begin{array}{rcl} x_1' &=& f_1(x_1, x_2) = g(x_1) - x_2\,, \\ x_2' &=& f_2(x_1, x_2) = \sigma x_1 - \gamma x_2\,, \end{array} \tag{8}$$

where σ and γ are positive constants and the function g is given by $g(x) = -x(x-1/2)(x-1)$.

(a) In the x_1-x_2 plane draw the graph of the curves $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$.

(b) Consider the rectangles $ABCD$ whose sides are parallel to the X_1 and x_2 axis with two opposite corners located on the $f_2(x_1, x_2) = 0$. Show that if the rectangle is taken sufficiently large, a solution which start inside the rectangle stays inside the rectangle forever. Deduce from this that the equations for any initial conditions x_0 have a unique solutions for all time $t > 0$.