Math 645: Problem Set 1

1. Determine whether the following sequences of functions are Cauchy sequences with respect to the uniform norm $\|\cdot\|_\infty$ on the given interval I. Determine the limit $f_n(x)$ if it exists.

   (a) $f_n(x) = \sin(2\pi nx)\,, \quad I = [0,1]$

   (b) $f_n(x) = \dfrac{x^n - 1}{x^n + 1}\,, \quad I = [-1,1]$.

   (c) $f_n(x) = \dfrac{1}{n^2 + x^2}\,, \quad I = [0,1]$

   (d) $f_n(x) = \dfrac{nx}{1+(nx)^2}\,, \quad I = [0,1]$

2. Show that $\|f\|_2$ is a norm on $\mathcal{C}([0,1])$.

3. (a) Let $f : U \to \mathbf{R}^n$ where $U \subset \mathbf{R}^n$ is an open set and suppose that f satisfies a Lipschitz condition on U. Show that f is uniformly continuous on U.

   (b) Let $f : E \to \mathbf{R}^n$ where $E \subset \mathbf{R}^n$ is a compact set. Suppose that f is locally Lipschitz on E, show that f satisfies a Lipschitz condition on E.

   (c) Show that $f(x) = 1/x$ is locally Lipschitz but that it does not satisfy a Lipschitz condition on $(0,1)$.

   (d) Show that $f(x) = \sqrt{|x|}$ is not locally Lipschitz.

   (e) Does the Cauchy problem $x' = 1/x$, $x(0) = a > 0$ have a unique solution? Solve it and determine the maximal interval of existence. What is the behavior of the solution at the boundary of this interval?

4. (a) Derive the following *error estimate* for the method of successive approximations. Let x be a fixed point given by this method. Show that

   $$\|x - x_k\| \leq \frac{\alpha}{1-\alpha}\|x_k - x_{k-1}\|\,, \tag{1}$$

   where α is the contraction rate.

   (b) Consider the function $f(x) = e^x/4$ on the interval $[0,1]$. Show that f has a fixed point on $[0,1]$. Do some iterations and estimate the error rigorously using (a).

5. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

   $$f(x) \;=\; \begin{cases} x + e^{-x/2} & \text{if } x \geq 0 \\ e^{x/2} & \text{if } x \leq 0 \end{cases}\,. \tag{2}$$

   (a) Show that $|f(x) - f(y)| < |x - y|$ for $x \neq y$.

   (b) Show that f does not have a fixed point.

   Explain why this does not contradict the Banach fixed point theorem.

6. Consider the IVP

   $$x' \;=\; x^3\,, \quad x(0) = a\,. \tag{3}$$

   (a) Apply the Picard-Lindelöf iteration to compute the first three iterations $x_1(t)$, $x_2(t)$, $x_3(t)$.

(b) Find the exact solution and expand it in a Taylor series around $t = 0$. Show that the first few terms agrees with the Picard iterates.

(c) How does the number of correct terms grow with iteration?

7. Apply the Picard-Lindelöf iteration to the Cauchy problem

$$x_1' = x_1 + 2x_2\,, \qquad x_1(0) = 0 \tag{4}$$

$$x_2' = t^2 + x_1\,, \qquad x_2(0) = 0 \tag{5}$$

Compute the first five terms in the taylor series of the solution.

8. (a) Let $I = [t_0 - \alpha, t_0 + \alpha]$ and for a positive constant κ define

$$\|x\|_\kappa = \sup_{t \in I} \|x(t)\| e^{-\kappa|t-t_0|}\,.$$

Show that $\|\cdot\|_\kappa$ defines a norm and that the space

$$E = \{x : I \to \mathbf{R}^n\,,\, x(t) \text{ continuous and } \|\mathrm{x}\|_\kappa < \infty\}$$

is a Banach space.

(b) Consider the IVP $x' = f(t,x)$, $x(t_0) = x_)$. Give a proof of Theorem 1.3.4 in the classnotes by applying the Banach fixed point theorem in the Banach space E with norm $\|\cdot\|_\kappa$ for a well-chosen κ.

(c) Suppose that $f(t,x)$ satisfy a *global Lipschitz condition*, i.e., there exists a positive $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbf{R}^n \text{ and for all } t \in \mathbf{R}\,. \tag{6}$$

Show that the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$ has a unique solution for all $t \in \mathbf{R}$. *Hint:* Use the norm defined in (a).

9. Consider the map T given by

$$T(f)(x) = \sin(2\pi x) + \lambda \int_{-1}^{1} \frac{f(y)}{1 + (x-y)^2}\, dy$$

(a) Show that if $f \in \mathcal{C}([-1,1], \mathbf{R})$ then so is $T(f)$.

(b) Find a λ_0 such that T is a contraction if $|\lambda| < \lambda_0$ and T is not a contraction if $|\lambda| > \lambda_0$. *Hint:* For the second part find a pair f, g such that $\|T(f) - T(g)\|_\infty > \|f - g\|_\infty$.

10. Let us consider $\mathbf{R}^2$ with the norm $\|x\| = \max\{|x_1|, |x_2|\}$. Let $f\,;\, \mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$f(x_1, x_2) = \begin{pmatrix} x_1^2 + 2x_2^2 + 5\cos(x_2) \\ 4x_1x_2 + 3 \end{pmatrix} \tag{7}$$

Let $K = \{(x_1, x_2)\,,\, |x_1| < 1, |x_2| \le 2\}$. Find an explicit Lipschitz constant L for f on K.

11. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class $\mathcal{C}^1$ and satisfy $f(0,0) = 0$. Suppose that $x(t)$ is a solution of the ODE

$$x'' = f(x, x')\,, \tag{8}$$

which is not identically 0. Show that $x(t)$ has simple zeros. Examples: the harmonic oscillator $x'' + x =$ or the mathematical pendulum $x'' + \sin(x) = 0$.