Math 624, Spring 2014: Problem set 6

1. Problem 10, p. 36

2. Problem 12 &13, p. 37

3. (a) Show that the space of Hölder continuous function (with $0 < \alpha \leq 1$)

   $$C^\alpha([a,b]) = \{f : [a,b] \to \mathbf{C}\,;\, |f(x) - f(y)| \leq M|x-y|^\alpha \text{ for all } x, y \in [a,b]\}$$

   with

   $$\|f\| = \sup_x |f(x)| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

   is a Banach space.

   (b) Let $(X, \mathcal{F})$ be a measurable space. Show that $M(X, \mathcal{F})$, the set of all signed finite measure on $(X, \mathcal{F})$ together with

   $$\|\mu\| = |\mu|(X)$$

   is a Banach space. (Here $|\mu|$ denotes the total variation of μ.)

4. Problem 30, p. 43

5. (a) Show that l^∞ is not separable.

   (b) To show that $(l^\infty)^* \neq l^1$, consider the subspace $c = \{x \in l^\infty\,;\, \lim_n x_n = x_\infty \text{ exists }\}$, define the functional $l(x) = x_\infty$ and use Hahn-Banach.

6. Let B be a Banach space and $\mathcal{L}(B)$ be the Banach space of all bounded operators $T : B \to B$. with $\|T\| = \sup_{v \in B, \|v\|=1} \|Tv\|$.

   (a) Let $I : B \to B$ denote the identity operator. Show that $\|I - T\| < 1$ then T is invertible.
   Hint: Show that $T^{-1} = \sum_{n=0}^\infty (I - T)^n$.

   (b) Show that the set of invertible operators is an open set in $\mathcal{L}(B)$.
   Hint: Given an invertible T let S be such that $\|S - T\| \leq \|T^{-1}\|^{-1}$ and use part (a).