Math 624, Spring 2014: Problem set 5

1. Let μ be a finite measure and let f, g be a bounded measurable function, i.e. $f, g \in L^\infty(\mu)$

   (a) Show that the map $\mathcal{F}(f) = \log \int \exp(f) d\mu$ is convex on $L^\infty(\mu)$ if and only if the map $G(t) = \log \int \exp(f + tg) d\mu$ is convex on $\mathbb{R}$ for any f, g.

   (b) Show that the convexity of F or G follows from Hölder's inequality.

   (c) Show that Jensen inequality implies that $G(t)$ is convex.

   (d) Show that the convexity of F implies Hölder's inequality. *Hint:* Consider first $F = e^f$ and $G = e^g$ for some $f, g \in L^\infty$ and then take a limit.

2. **Generalized Hölder's inequality.** Let $1 \leq p_j \leq \infty$ for $j = 1, \cdots, n$ and suppose

$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \leq 1 \,.$$

   Show that if $f_j \in L^{p_j}$ then $\prod_{j=1}^n f_j \in L^r$ and

$$\left\| \prod_{j=1} f_{p_j} \right\|_r \leq \prod_{j=1}^{n} \|f_j\|_{p_j} \,.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f, g be nonnegative functions. Suppose that $0 < p < 1$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$\int fg d\mu \geq \left(\int f^p \, d\mu \right)^{1/p} \left(\int g^q \, d\mu \right)^{1/q} \,.$$

   Hint: use Hölder's inequality for suitable functions.

4. Let $(X, \mathcal{M}, \mu)$ and suppose that $f \in L^1(\mu) \cap L^2(\mu)$. Prove that

$$\lim_{p \to 1+} \|f\|_p = \|f\|_1 \,.$$

5. Let $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$ and suppose $f \in L^\infty(\mu)$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

6. **Roger's inequality** Let f, g be positive measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Let $0 < t < r < m < \infty$.

(a) Show that if the integrals on the right are finite then the following holds (Roger's inequality)

$$\left(\int fg^r \, d\mu\right)^{m-t} \leq \left(\int fg^t \, d\mu\right)^{m-r} \left(\int fg^m \, d\mu\right)^{r-t}$$

Hint: Use Hölder's inequality.

(b) Show conversely how the Hölder inequality follows from the Roger's inequality. *Hint:* let $t = 1$ amd $m = 2$.