Math 624, Spring 2014: Problem set 4

1. Problem 24, p. 150

2. Problem 16, p. 148
   Hint: For (a) use corollary 3.7. For (b) write $F' = g + h$ where g is a step function and h has small L^1 norm.

3. (Integration by parts) Let F and G be function of bounded variation and right-continuous functions on [a,b] and let μ_F and μ_G be the corresponding signed Borel measures (recall that we have these measures are uniquely determined by $\mu_F((c,d]) = F(d) - F(c)$).

   (a) Show that if either F or G is continuous then we have the "integration by parts" formula
   $$\int_{(a,b]} F d\mu_G + \int_{(a,b]} G d\mu_F \;=\; F(b)G(b) - F(a)G(a) \tag{1}$$
   Hint: Assume G is continuous and compute $\mu_F \times \mu_G(A)$ where $A = \{a < x \leq y \leq b\}$; use Fubini.

   (b) By modifying the argument in (a) show that we have
   $$\int_{[a,b]} \frac{F(x) + F(x-)}{2} d\mu_G + \int_{[a,b]} \frac{G(x) + G(x-)}{2} d\mu_F \;=\; F(b)G(b) - F(a-)G(a-) \tag{2}$$

   (c) Show that if F and G are absolutely continuous we have
   $$\int_a^b FG' dx + \int_a^b GF' dx \;=\; F(b)G(b) - F(a)G(a) \tag{3}$$

4. Let F be of bounded variation and right-continuous and μ_F is the corresponding Borel signed measure. Show that the total variation of μ_F satisfies
   $$|\mu_F| = \mu_{TF}$$
   where $TF(x)$ is the total variation of F.

5. Suppose $\{F_j\}$ is a sequence of nonnegative increasing functions on $[a,b]$ such that $F(x) = \sum_j F_j(x) < \infty$ for all $x \in [a,b]$. Show that $F' = \sum F_j'$ for a.e. x.
   Hint: WLOG you may assume that F_j are right-continuous and consider then the corresponding Borel measures .

6. Consider the Cantor-Lebesgue function $F(x)$ on [0,1] and let $\{[a_n, b_n]\}$ be an enumeration of all intervals in $[0,1]$ with rational endpoints. Let $G(x) = \sum_n \frac{1}{2^n} F\left(\frac{x-a_n}{b_n-a_n}\right)$. Show that G is continuous, strictly increasing on $[0,1]$ and $G'(x) = 0$ for a.e. x.
   Hint: Use problem 5.