Math 624, Spring 2014: Problem set 2

1. (a) Consider a function $f \in L^2([-\pi,\pi])$ with Fourier series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ and partial sum $S_n(f) = \sum_{|n|\le N} c_n e^{inx}$. Show that for any interval $[a,b] \subset [-\pi,\pi]$ we have

$$\int_a^b f(x)\,dx = \sum_{n\in\mathbb{Z}} \int_a^b c_n e^{inx}\,dx\,.$$

In particular if $f \in L^2$ and $g(x) = \int_a^x f(t)dt$ then the Fourier series of $g(x)$ for every x and the Fourier coefficients of g can be obtained from the Fourier coefficients of f.

(b) Consider the the function $f(x) = x/2$ on the interval $[-\pi,\pi]$. Show that the Fourier series of f is

$$\sum_{n\in\mathbb{Z}} (-1)^{n+1}\frac{\sin(nx)}{n}$$

(c) Use part (a) twice to show the identities

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

2. Suppose that f is periodic of period 2π. For $0 < \alpha \le 1$ we say that f is $C^{0,\alpha}$ is Hölder continuous with exponent α, i.e. we have $|f(x) - f(y)| \le C|x-y|^\alpha$ with $0 < \alpha \le 1$. For positive integer k we say that f is if class $C^{k,0}$ is f is k-times continuously differentiable and we say that f is if class $C^{k,\alpha}$ is f is k- times differentiable and its k^{th} derivative is Hölder continuous with exponent α.

Show that if $f \in C^{k,\alpha}$ then its Fourier coefficients satisfy

$$|c_n| \le \frac{1}{|n|^{k+\alpha}}$$

Hint: For $C^{0,\alpha}$ in the integral expression for c_n do the substitution $x \to x + \frac{\pi}{n}$. Integral by parts.

3. Consider a function $f \in L^2([-\pi,\pi])$ with Fourier coefficients c_n and Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}\,, \quad c_n = \frac{1}{2\pi}\int f(x)e^{-inx}\,dx$$

(a) Show that if f is real-valued then its Fourier series can be written as

$$\frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

for suitable real coefficients a_n, b_n. What happens to the coefficents a_n, b_n, if f is even, respectively odd?

(b) Prove that one can write for any $0 \leq x \leq \pi$

$$\sin(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for suitable coefficients a_n which you will compute. Justify the covergence in the formula.
Hint: Note that the formula holds for $0 \leq x \leq \pi$ not $-\pi \leq x \leq \pi$. Extend the function $\sin(x)$ on $0 \leq x \leq \pi$ to an even function on $[0, 2\pi]$.

4. Problem 4, p. 146

5. Problem 5, p. 146

6. Problem 9, p. 195