Math 624, Spring 2014: Problem set 1

1. Show that if f and g belong to $L^1(\mathbf{R}^d)$ then we have

$$\widehat{f \star g}(\xi) \;=\; \hat{f}(\xi)\,\hat{g}(\xi)$$

   where $f \star g$ denote the convolution product.

2. Problem 22, p. 94. This shows that the Fourier transform maps L^1 functions into functions which are bounded, continuous, and converges to 0 as $|\xi| \to \infty$.

3. Problem 23, p. 94,

4. Problem 25, p. 95 This show that the Fourier transform does NOT map L^1 into itself.

5. Show that $c_n \left[\cos(\frac{\pi x}{2})\right]^n$, $-1 \le x \le 1$, for suitable constants n is an approximation of the identity. Use this and suitable trigonometric formulas to prove a version of the Weierstrass approximation theorem for periodic continuous function on $[-\pi, \pi]$ by trigonometric polynomials.

6. Let $f \in L^1(\mathbf{R})$. Use the Fourier transform to solve the equation

$$u(x) - \frac{d^2}{dx^2}u(x) \;=\; f(x)$$

   Hint: It going to be useful to know what is the Fourier transform of $e^{-a|x|}$ for $a > 0$, so compute it!

7. Use Fourier series solve the wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) \;=\; 0\,, \quad u(0,x) = f(x)\,, \quad \frac{\partial}{\partial t}u(0,x) = g(x)\,,$$

   where $x \in \mathbf{R}$, $t \in [0, \infty)$. What are reasonable assumptions on f and g?