Math 624: Homework 7

1. (a) Show the following generalization of Chebyshev inequality. Let $1 \leq p < \infty$ and let $f \in L^p(X, \mu)$. Then for any $a > 0$

$$\mu\{x \in X\,;\, |f(x)| > a\} \leq \frac{1}{a^p} \int |f|^p \, d\mu\,.$$

   (b) Let $1 \leq p < \infty$ and let $\{f_n\}$ be a convergent sequence in $L^p(X, \mu)$, i.e., there is $f \in L^p(X, \mu)$ so that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that f_n converges in measure to f.
   Hint: Use (a)

   (c) Let $1 \leq p < \infty$ and let $\{f_n\}$ be a convergent sequence in $L^p(X, \mu)$, i.e., there is $f \in L^p(X, \mu)$ so that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that there exists a subsequence $\{f_{n_k}\}$ which converges almost everywhere.

   Hint: Go back look at Chapters 1 and 2!

2. Show that for $1 \leq p < \infty$ the Banach space l^p is separable but that l^∞ is not separable.
   Hint: For l^∞ you can either argue by contradiction or prove that l^∞ contains an uncountable sets of elements so that any pair of elements in that set are at distance at least 1.

3. Let m denote Lebesgue measure and let $[a, b]$ be an interval with $-\infty \leq a < b \leq \infty$. Show that $L^p([a, b], m)$ is separable and that the continuous functions with compact support are dense in $L^p([a, b], m)$. Show $L^\infty([a, b], m)$ is not separable.
   Hint: Show that L^∞ contains an uncountable sets of elements so that any pair of elements in that set are at distance at least 1.

4. Let h be a real number and let f be a measurable function defined on $\mathbf{R}$. Let $T_h(f) = f(t + h)$, i.e. T_h is the action of translation on measurable functions. Show

   (a) If $1 \leq p \leq \infty$, $\|T_h f\|_p = \|f\|_p$, i.e. T_h is an isometry.

   (b) If $1 \leq p < \infty$, $\lim_{h\to 0} \|T_h f - f\|_p = 0$, i.e. the translations act continuously on $L^p(\mathbf{R}, m)$.
   Hint: Prove it first for continuous functions with compact support.

(c) $\|T_h f - f\|_\infty$ does not necessarily converge to 0 as h goes to 0, i.e., the translations do not act continuously on $L^\infty(\mathbf{R}, m)$.

5. Show the Banach space l^∞ and $L^\infty([0,1], m)$ where m is Lebesgue measure are *not* separable.
Hint: For l^∞ you can prove it by contradiction. For both l^∞ and $L^\infty([a,b], m)$ you can prove it by constructing an uncoutable set of elements all of which are at distance 1 from all others.

6. Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and $f \in L^1(\mu) \cap L^\infty(\mu)$. Show that
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty .$$
Hint: It is easy to see that $\lim_{p\to\infty} \|f\|_p \le \|f\|_\infty$. To show the reverse inequality, let $\epsilon > 0$, then $E = \{x\,;\, |f(x)| > \|f\|_\infty - \epsilon\}$ has positive measure and $(\|f\|_\infty - \epsilon)\chi_E \le |f|$.

7. **Generalized Hölder's inequality.** Let $1 \le p_j \le \infty$ for $j = 1, \cdots, n$ and suppose
$$\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r} \le 1 .$$
Show that if $f_j \in L^{p_j}$ then $\prod_{j=1}^n f_j \in L^r$ and
$$\left\| \prod_{j=1} f_{p_j} \right\|_r \le \prod_{j=1}^n \|f_j\|_{p_j} .$$
Hint: Start with $n = 2$.

8. **Integral Operators** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $K(\cdot,\cdot)$ be a measurable function on $X \times X$ equipped with the product σ-algebra. Suppose that there exists a constant C such that
$$\int |K(x,y)| d\mu(x) \;\le\; C \text{ for a.e } y\,,$$
and
$$\int |K(x,y)| d\mu(y) \;\le\; C \text{ for a.e } y\,.$$
Let $1 \le p \le \infty$ and $f \in L^p(\mu)$. Show

(a) The integral

$$Tf(x) = \int K(x,y)f(y)d\mu(y)$$

converges absolutely for a.e. x, i.e. $\int |K(x,y)f(y)|d\mu(y) < \infty$ for a.e. x.
Hint: Apply Hölder to $|K(x,y)f(y)| = |K(x,y)|^{1/q}\left(|K(x,y)|^{1/p}|f(y)|\right)$

(b) The function Tf defined in (a) is in $L^p(\mu)$ and $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(\mu)$.

(c) Suppose $1 \leq p < \infty$. Compute the adjoint operator T^*.