Math 624: Homework 6

1. On the Banach space $\mathbf{C}^d$ consider the norm $\|x\|_1 = \sum_{i=1}^d |x_i|$, $\|x\|_2 = (\sum_{i=1}^d |x_i|^2)^{1/2}$, and $\|x\|_\infty = \sup_{1\le i\le d} |x_i|$. Consider a linear map $A : \mathbf{C}^d \to \mathbf{C}^d$, i.e., a $n\times n$ matrix $A = (a_{ij})_{i,j=1}^d$. For $p = 1, 2, \infty$ define

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p},$$

   i.e., $\|A\|_p$ is the operator norm of A as an operator on the Banach $\mathbf{C}^n$ equipped with $\|\cdot\|_p$. Show that

$$\|A\|_1 = \max_j \sum_i |a_{ij}| \quad \|A\|_\infty = \max_i \sum_j |a_{ij}|.$$

   and that $\|A\|_2$ is the square root of the largest eigenvalue of A^*A.

2. Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all bounded operator from X to X.

   (a) Let $T \in \mathcal{L}(X)$ and suppose that $\|I - T\| < 1$ (I is the identity, i.e. $Ix = x$). Show that T is invertible with inverse $T^{-1} = \sum_{n=0}^\infty (I-T)^n$.

   (b) Let $T \in \mathcal{L}(X)$ be invertible and suppose that $S \in \mathcal{L}(X)$ satisfies $\|S - T\| \le \|T^{-1}\|^{-1}$, then S is invertible.
   Remark: This shows that the set of invertible operators is an open subset of $\mathcal{L}(X)$.

3. Let $C([0,1])$ be the the Banach space of continuous functions on $[0,1]$ equipped with the uniform norm $\|f\|_u = \sup_{t\in[0,1]} |f(t)|$.

   (a) Let $k \in C([0,1]\times[0,1])$ be a continuous function. Define $A : C([0,1]) \to C([0,1])$ by

$$Af(t) = \int_0^1 k(t,s) f(s)\, ds.$$

   Show that A is a bounded operator and $\|A\| = \max_{t\in[0,1]} \int_0^1 |k(t,s)|\, ds$.

(b) Let $k \in C([0,1] \times [0,1])$ and $g \in C([0,1])$be given. Show that the integral equation (f is the unknown)

$$f(t) = g(t) + \int_0^1 k(t,s) f(s) ds \, ,$$

has a unique solution in $C([0,1])$ if $\int_0^1 |k(t,s)| \, ds < 1$.

(c) Write the solution in the form $f(t) = g(t) + \int_0^1 r(t,s) g(s) \, ds$ and determine $r(t,s)$.

Hint: For (b) and (c) use Exercise 2 (a).

4. Let k be an integer and let $C^k([0,1])$ denote the space of functions which have continuous derivatives up to order k, including one-sided derivatives at the endpoints.

   (a) Show that if $f \in C[0,1])$, then $f \in C^k([0,1])$ if and only if f is k-times continuously differentiable on $(0,1)$ and $\lim_{x \to 0+} f^{(j)}(x)$ $\lim_{x \to 1-} f^{(j)}(x)$ exists for $j \le k$. (Use the mean-value Theorem).

   (b) Show that if $\{f_n\}$ is a sequence in $C^1([0,1])$ such that $f_n \to f$ uniformly and $f_n' \to g$ uniformly then $f \in C^1([0,1])$ and $f' = g$. (Show that $f(x) - f(y) = \int_y^x g(s) ds$.)

   (c) Show that $\|f\| \equiv \sum_{j=0}^k \|f^{(j)}\|_u$ is a norm on $C^k([0,1])$ and that, with this norm $C^k([0,1])$ is a Banach space. (Use (b) and induction on k).

5. Consider the operator A which maps $f(t)$ to its derivative $f'(t)$. Show that $A : C^1([0,1]) \to C([0,1])$ is a bounded operator but that $A : C^1([0,1]) \to C^1([0,1])$ is not bounded.

6. Let $0 < \alpha \le 1$ and let H_α denote the space of all functions satisfying a Lipschitz condition with exponent α, i.e., if $f \in H_\alpha$ there exists a constant M such that $|f(x) - f(y)| \le M|x-y|^\alpha$ for all $x, y \in [0,1]$. Define

$$\|f\|_{H_\alpha} \equiv \sup_{x \in [0,1]} |f(x)| + \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{x - y|^\alpha} \, .$$

   Show that H_α with $\| \cdot \|_{H_\alpha}$ is a Banach space.

7. Let X be a normed vector space and let M be a closed subspace of X. Define an equivalence relation $x \sim y$ if $x-y \in \mathcal{M}$. The equivalence class of x is denoted by $x+M$ and the set of equivalence classes is denoted by X/M which is a vector space with the operations $(x+M)+(y+M) = (x+y)+M$ and $\alpha(x+M) = \alpha x + M$.

   (a) Show that $\|x+M\| \equiv \inf\{\|x+y\| \; : \; y \in M\}$ is a norm on the quotient space X/M

   (b) If X is complete then X/M is complete.
   Hint: Use the criteria for completeness in terms of absolutely convergent sequences.

8. **The adjoint of an operator**. Suppose X and Y are Banach spaces and $T : X \to Y$ is a bounded operator. We define the *adjoint of* T, T^* as the operator $T^* : Y^* \to X^*$ given by

$$T^* f(x) = f(Tx)$$

   where $f \in Y^*$ and $x \in X$.

   (a) Show that T^* is a well defined bounded operator and $\|T^*\| = \|T\|$.

   (b) Verify that if $X = Y$ and is a Hilbert space this definition coincides with the definition given in Chapter 4.

   (c) Fix some $g \in X^*$ and some $u \in Y$ and define $S : X \to Y$ by $S(x) = g(x)u$. Show that S is bounded and $\|S\| = \|g\|\|u\|$. Compute the adjoint S^* of S.