Math 624: Homework 4

1. Exercise 8, p.313.

2. Exercise 10 p. 314

3. Suppose ν_i is a σ-finite signed measure and μ_i a σ-finite measures on the the measure space $(X_i, \mathcal{M}_i)$, for $i = 1, 2$. Show that if $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)\,.$$

4. Let $[0, 1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0, 1]$. Let m denote the Lebesgue measure on $[0, 1]$ and let μ denote the counting measure on $[0, 1]$. Show that $m \ll \mu$ but that there exists no f such that $dm = f d\mu$.

5. Exercise 11, p. 314.

6. Suppose that F and G are complex-valued function of bounded variations on $[a, b]$.

   (a) Show the following integration by parts formula for Lebesgue-Stieljes integrals: If at least one of F and G is continuous then

   $$\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a)$$

   Hint: Without restriction of generality you can assume that F and G are increasing and that G is continuous. Let $\Omega = \{(x, y) : a < x \le y \le b\}$ and compute $\mu_F \times \mu_G(\Omega)$ (in two ways) using Fubini.

   (b) If one does not assume that F or G or continuous then show that

   $$\int_{[a,b]} \frac{F(x) + F(x-)}{2} dG(x) + \int_{[a,b]} \frac{G(x) + G(x-)}{2} dF(x)$$
   $$= F(b)G(b) - F(a-)G(a-)\,. \qquad (1)$$

7. Suppose G is a continuous increasing function on $[a, b]$ and let $c = G(a)$ and $d = G(b)$.

(a) Show that if $E \subset [c,d]$ is a Borel set, then $m(E) = \mu_G(G^{-1}(E))$. (Prove it first for an interval.)

(b) Show the following change of variables formula. If f is Borel measurable and integrable on $[c,d]$ then $\int_{[c,d]} f dx = \int_{[a,b]} f(G(x)) dG(x)$.

(c) Show that the formula in (b) might fail if G is merely right-continuous.

8. Let μ be a signed or complex measure on $(X, \mathcal{M})$. Show that for any $E \in \mathcal{M}$

$$
\begin{aligned}
&|\nu|(E)| \\
&= \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| \;:\; n \geq 1, E_1, \cdots E_n \text{ disjoint}, E = \bigcup_{j=1}^{n} E_j \right\} \quad (2) \\
&= \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \;:\; E_1, E_2, \cdots \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j \right\} \quad (3) \\
&= \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\}. \quad (4)
\end{aligned}
$$

Hint: Prove that (2) $\leq$ (3) $\leq$ (4) $\leq$ (2).

9. Let F be of bounded variation on $[a,b]$ and let $G(x) = |\mu_F|([a,x])$. Show that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$. To do this prove

(a) From the definition of T_F, we have $T_F \leq G$.

(b) $|\mu_F(E)| \leq \mu_{T_F(E)}$ for any Borel set E. (Consider first an interval.)

(c) Using the previous problem we have $|\mu_F| \leq \mu_{T_F}$ and hence $G \leq T_F$.