Math 624: Homework 3

1. Prove Proposition 1.6 in Chaper 6 of the textbook.

2. Let $[0,1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Let m denote the Lebesgue measure on $[0,1]$ and let μ denote the counting measure on $[0,1]$, i.e. for $E \subset [0,1]$, $\mu(E)$ is the number of elements in E. Let $D = \{(x,x)\,,\, x \in [0,1]\}$ denote the diagonal in $[0,1] \times [0,1]$. Show that $\int\int \chi_D dm d\mu$, $\int\int \chi_D d\mu dm$, and $\int \chi_D d(m \times \mu)$ are all unequal. Explain why this does not contradict Fubini Theorem.

   Hint: To compute $\int \chi_D d(m \times \mu)$ go back to the definition of $m \times \mu$.

3. Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space. Let Y be a countable set, $\mathcal{N} = \mathcal{P}(Y)$ and let ν be any σ-finite measure on Y, e.g., the counting measure. Show that the Fubini-Tonelli Theorem is valid in this case.

4. Exercise 14, p. 315

5. Let $(X_i, \mathcal{M}_i, \mu_i)$, $i = 1, 2, 3, \cdots$ be a countable collection of *finite* measure spaces *with* $\mu_i(X_i) = 1$. Consider the Cartesian product $X = \prod_{i=1}^{\infty} X_i$. Each point in X is represented by a sequence $x = \{x_i\}$ with $x_i \in X_i$. We say that the set E is a cylinder if E has the form

   $$E = \{x = \{x_i\}\,;\, x_i \in E_i \in \mathcal{M}_i \text{ and } E_i = X_i \text{ for all but finitely many } i\}\,.$$

   For a cylinder set define $\mu_0(E) = \prod_{j=1}^{\infty} \mu_i(E_i)$. It is possible to show that μ_0 extends to a unique finite measure on X, μ, on X which is called the product measure. The proof of this fact is quite technical and you can simply accept it here (see the probability class). Such spaces are a source of good examples: do Exercise 23, (a) and (b), p. 318.

6. Exercise 12, p. 315 (we need this for the constructon of polar coordinates).

7. Exercise 5, p. 313. Deduce from this fact the amusing fact that the volume of the d-dimensional ball of radius 1 tends to 0 as $d \to \infty$. Recall that the Gamma function $\Gamma(x)$ is given, for $x \geq 0$, by

   $$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$

and that by integration parts you can prove that $\Gamma(x+1) = x\Gamma(x)$.

8. Let $f(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ where α_i, $i = 1, \cdots, n$ are nonnegative integers, i.e. $f(x)$ is a monomial. Proceeding as in the previous show that $\int f d\sigma = 0$ if any α_j is odd, and if all α_j's are even then, with $\beta_j = \frac{\alpha_j+1}{2}$,

$$\int f \, d\sigma = \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1 + \beta_n)}.$$