Math 624: Homework 2

1. Exercise 1, p.312.

2. Exercise 2, p.312.

3. Exercise 3, p.312.

4. Let $\mu_\star$ be an outer measure. Show that if E is Carathéodory measurable and if A is an arbitrary subset of X we have

$$\mu_\star(E \cup A) + \mu_\star(E \cap A) \;=\; \mu_\star(E) + \mu_\star(A)\,.$$

5. A set function μ defined on a σ-algebra $\mathcal{M}$ of subsets of X is called a *finitely additive measure* if $\mu(\emptyset) = 0$ and if for any finite collection of pairwise disjoint sets $E_1, \cdots, E_n$ in $\mathcal{M}$ we have $\mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$.

   Suppose $\mu(X) < \infty$. Show that a finitely additive measure μ is a measure (i.e., it is countable additive) if and only if μ is "continuous at $\emptyset$", that is for any decreasing sequence of sets $A_1 \supset A_2 \supset \cdots$ with $A_i \in \mathcal{M}$ and $\cap_{n=1}^\infty A_n = \emptyset$ one has $\lim_{n\to\infty} \mu(A_n) = 0$.

   Hint: To show that the continuity at $\emptyset$ is sufficient, set $F = \cup_i E_i$ and $A_n = F \setminus \cup_{i=1}^{n-1} E_i$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. For any set $E \subset X$ let us define

$$\mu_\star(E) \;=\; \inf\left\{ \sum_{n=1}^\infty \mu(A_n)\,;\; A_n \in \mathcal{M}, E \subset \bigcup_{i=1}^\infty A_i \right\}\,.$$

   Show

   (a) $\mu_\star$ is an outer measure.

   (b) For any $A \in \mathcal{M}$, $\mu_\star(A) = \mu(A)$.