Math 624: Homework 1

1. Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot, \cdot)$. By definition we say that x_n converges to x in $\mathcal{H}$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. Furthermore we say that x_n converges to x *weakly* in $\mathcal{H}$ if $\lim_{n\to\infty}(x_n, y) = (x, y)$ for all $y \in \mathcal{H}$.

   (a) Show that if x_n converges to x then x_n converges to x weakly.

   (b) By giving a counterexample, show that weak convergence does not imply convergence.

   (c) Show that x_n converges to x in $\mathcal{H}$ if and only if x_n converges to x weakly in $\mathcal{H}$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$.

   (d) Let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}$. Show that x_n converges to x weakly in $\mathcal{H}$ if and only if $\lim_{n\to\infty}(x_n, e_k) = (x_n, e_k)$ for all k.

   (e) As we have seen in class, if $\{x_n\}$ is a sequence with $\|x_n\| \leq 1$ then $\{x_n\}$ does not necessarily has a convergent subsequence (for example take $x_n = e_n$). However one can show that $\{x_n\}$ has a subsequence which converges weakly to some x. In other terms the unit ball of $\mathcal{H}$ is not compact for the topology induced by the norm $\|\cdot\|$ but it is compact for the weak topology induced by the weak convergence.

   Hint: The proof of this fact use a trick known as "Cantor diagonal argument" and it works as follows. Consider the sequence of complex numbers (x_n, e_1). Show that there exists a subsequence of x_n, call it x_{n1}, such that (x_{n1}, e_1) is convergent. Consider next the sequence (x_{n1}, e_2) and show there is a subsequence x_{n2} such that (x_{n2}, e_1) and (x_{n2}, e_2) converge. By repeating this indefinitely one obtains subsequence x_{nk}. Finally consider the diagonal sequence x_{nn} (hence the name of the trick) and show that (x_{nn}, e_k) converges for all k.

2. Let $\mathcal{H}$ be a Hilbert space and let T_n and T be bounded linear operators on $\mathcal{H}$. There are several ways to define the convergence of the sequence $\{T_n\}$ to T.

   - T_n converges to T *in norm* if $\lim_{n\to\infty} \|T_n - T\| = 0$.

- T_n converges to T *strongly* if $T_n f$ converges to Tf in $\mathcal{H}$ for all $f \in \mathcal{H}$, i.e. $\lim_{n\to\infty} \|T_n f - Tf\| = 0$ for all $f \in \mathcal{H}$.
- T_n converges to T *weakly* if $T_n f$ converges to Tf weakly in $\mathcal{H}$ for all $f \in \mathcal{H}$, i.e., $\lim_{n\to\infty}(T_n f, g) = (Tf, g)$, for all $f, g \in \mathcal{H}$.

(a) Show that convergence in norm implies strong convergence which itself implies weak convergence.

(b) Show, by examples, that weak convergence does not imply strong convergence and that strong convergence does not imply convergence in norm.

3. Let $f \in L^1(\mathbf{R})$. Use the Fourier transform to solve the equation

$$u(x) - \frac{d^2}{dx^2}u(x) = f(x)$$

Hint: It is useful to remember what is the Fourier transform of $e^{-a|x|}$.

4. Use Fourier series solve the wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = 0\,, \quad u(0,x) = f(x)\,, \quad \frac{\partial}{\partial t}u(0,x) = g(x)\,,$$

where $x \in \mathbf{R}$, $t \in [0, \infty)$ and f, g and $u(t, \cdot)$ are periodic functions of x of period 2π.

5. **Abel summability**

(a) Given a sequence $\{a_k\}_{k\geq 0}$ and $N \geq M$ define $S_M^N = \sum_{k=M}^{N} a_n$. Show the following formula which is known as integration by parts for series (why the name?)

$$\sum_{k=M}^{N} a_k b_k = \sum_{k=M}^{N-1} S_M^k (b_k - b_{k+1}) + S_M^N b_N\,.$$

(b) Suppose $\sum_{n=0}^{\infty} a_n$ is convergent and $0 \leq r \leq 1$. Show that

$$\left|\sum_{k=M}^{N} r^k a_k\right| \leq \sup_{J \geq M} |S_M^J|\,.$$

(c) Show that the series $S(r) = \sum_{k=0}^{\infty} r^k a_k$ is uniformly convergent for $0 \le r \le 1$ and that $\lim_{r\to 1-} S(r) = \sum_{k=0}^{\infty} a_k$.

6. **Cesaro summability** Let $\{a_n\}_{n\ge 0}$ be a sequence and set

$$b_m = \frac{1}{m+1}(a_0 + a_1 + \cdots + a_m)\,.$$

Show that if $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = a$ then $\{b_m\}$ converges and $\lim_{m\to\infty} b_m = a$. Note that the converse statement is not true, in general.

7. Let $f \in L^1([-\pi\,,\,\pi])$ with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$. As we have seen in class we can write the partial sums as

$$S_N f(x) \;=\; \sum_{n=-N}^{N} a_n e^{inx} \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) D_N(x-y)dy$$

where $D_N(x)$ is the Dirichlet kernel given by

$$D_N(x) \;=\; \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\,.$$

As we have seen in class $D_N(x)$ is *not* a good kernel. Set

$$K_N(x) \;=\; \frac{1}{N+1}\left(D_0(x) + \cdots + D_N(x)\right)\,,$$

$K_N(x)$ is known as the Fejér kernel. Show that

$$K_N(x) \;=\; \frac{1}{N+1}\left[\frac{\sin\frac{1}{2}(N+1)x}{\sin\frac{1}{2}x}\right]^2\,.$$

and verify that $K_N(x)$ is an approximation of the identity ($K_N(x) = 0$ if $|x| > \pi$). Conclude that

$$\sigma_N f(x) \equiv \frac{1}{N+1}\left(S_0 f(x) + \cdots + S_N f(x)\right)$$

converges to $f(x)$ for almost every x.