Some fact about $\sup$**,** $\inf$**,** $\limsup$ **and** $\liminf$

1 Supremum and Infimum

For a set X of real numbers, the number $\xi = \sup X$, the *supremum of* X (or *least upper bound of* X) is defined by

1. For all $x \in X$, $x \leq \xi$
2. For any $\epsilon > 0$ there exists x such that $x > \xi - \epsilon$.

and the infimum of X $\inf X$ is defined similarly.

2 Sequences, accumulation points, $\limsup$ and $\liminf$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. A point x is called an *accumulation point* of if there exists a subsequence $\{x_{n_k}\}$ which converges to x. A well-known theorem is

Theorem 2.1 Boltzano-Weierstrass Theorem *Any sequence which is bounded above (i.e there exists* M *such that* $x_n \leq M$ *for all* n*) has at least one accumulation point.*

Note by the way that this theorem is what is needed to prove that compact sets in $\mathbf{R}$ or $\mathbf{R}^d$ are exactly the sets which are closed and bounded.

It is instructive to have a look at the proof.

Proof: Consider the set

$$X = \{x\,;\ \text{infinitely many } x_n \text{ are } > x\}\,.$$

This set is bounded and we define $\xi = \sup X$. The number ξ is finite and by definition for any $\epsilon > 0$ only finitely many x_n satisfy $x_n \geq \xi + \epsilon$ and infinitely many x_n such that $x_n \geq \xi - \epsilon$. Therefore there are infinitely many x_n in the interval $[\xi - \epsilon\,,\, \xi + \epsilon]$.

For any integer k take $\epsilon = \frac{1}{k}$, one construct the subsequence in the following inductive way. Choose n_1 such $x_{n_1} \in [\xi - 1\,,\, \xi + 1]$, then inductively choose $n_k > n_{k-1}$ such that $x_{n_k} \in [\xi - 1/k\,,\, \xi + 1/k]$. The sequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to ξ. ■

This proof exhibits not any accumulation point, but the *largest accumulation point* and it is called the limit superior of the sequence $\{x_n\}$. We denote it by

$$\xi = \limsup_{n\to\infty} x_n = \sup\{x\,;\ \text{infinitely many } x_n \text{ are } > x\}\ .$$

Using sequences which are bounded below and the inf instead of the sup one defines

$$\xi = \liminf_{n\to\infty} x_n = \inf\{x\,;\ \text{infinitely many } x_n \text{ are } < x\}\ .$$

which is the smallest accumulation point of the sequence $\{x_n\}$.

3 Properties of $\limsup$ and $\liminf$

Trivially we have

$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$$

and

$$\lim_{n\to\infty} x_n = x \quad \text{if and only if} \quad \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n\,.$$

We also have

$$\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$$

$$\liminf(x_n + y_n) \ge \liminf x_n + \liminf y_n$$

You should prove this and note that the inequalities can be strict (Find such examples).

The $\limsup$ and $\liminf$ can also be written as follows

$$\limsup_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \sup_{k\ge n} x_k \;=\; \inf_{n\ge1} \sup_{k\ge n} x_k\,.$$

$$\liminf_{n\to\infty} x_n \;=\; \lim_{n\to\infty} \inf_{k\ge n} x_k \;=\; \sup_{n\ge1} \inf_{k\ge n} x_k\,.$$

We prove this for $\limsup$. Note first that $y_n \equiv \sup_{k\ge n} x_k$ is a decreasing sequence, i.e., $y_{n+1} \le y_n$, and thus it is convergent and we have $\lim_{n\to\infty} y_n = \inf_{n\to\infty} y_n$. Let ξ denote this limit, then for any $\epsilon > 0$ there exists N such that for all $n \ge N$ we have

$$\xi \le y_n = \sup_{k\ge n} x_k \le \xi + \epsilon\,.$$

Using the right inequality for $n = N$ shows that at most finitely many x_j are bigger than $\xi + \epsilon$. On the other using the definition for the sup and left inequality for $n = N$ we can find $n_1 \ge N$ such that $x_{n_1} > \xi - \epsilon$. Using the left inequality for $n_1 + 1$ we can find n_2 such that $x_{n_2} > \xi - \epsilon$ and thus there exists infinitely many x_j bigger than $\xi - \epsilon$. This shows that $\xi = \limsup_{n\to\infty} x_n$.