Math 623, Fall 2013: Problem set 9

1. Exercise 2, p. 312.
2. Exercise 3, p. 312.
3. Exercise 5, p. 313.
4. Exercise 8, p. 313
5. Exercise 10, p. 314
6. Consider the function

$$F(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ x^2+1 & \text{if } \ 0 \le x \le 2 \\ 9-x & \text{if } \ 2 \le x < 4 \\ 10 & \text{if } \ 4 \le x \end{cases}$$

   and consider the (signed) Borel measure $\mu = \mu_F$.

   (a) Compute the total variation $|\mu|$ and the positive and negative variations μ_+ and μ_-.

   (b) Compute $\int_{\mathbf{R}} x^2 d\mu$.

   (c) Let m denote the Lebesgue measure on $\mathbf{R}$. Compute the Lebesgue decomposition $\mu = \mu_a + \mu_s$ of μ with respect to m.

7. Exercise 14, p.315
8. Let $[0,1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Let m denote the Lebesgue measure on $[0,1]$ and let μ denote the counting measure on $[0,1]$, i.e., for $E \subset [0,1]$, $\mu(E)$ is the number of elements in E. Let $D = \{(x,x)\,,\, x \in [0,1]\}$ denote the diagonal in $[0,1] \times [0,1]$. Show that $\int\int \chi_D dm d\mu$, $\int\int \chi_D d\mu dm$, and $\int \chi_D d(m \times \mu)$ are all unequal. Why does this not contradict Fubini Theorem?

   Hint: To compute $\int \chi_D d(m \times \mu)$ go back to the definition of $m \times \mu$.

9. Suppose ν is a σ-finite signed measure and μ and λ are σ-finite (positive) measures on $(X, \mathcal{M})$.

   (a) Suppose that $\nu \ll \mu$. Show that if $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(b) Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Show that $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}.$$

Hint: Use (a). This equality is called the *chain rule* (why?).

(c) Suppose $\mu \ll \lambda$ and $\lambda \ll \mu$. Show that

$$\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1$$

for μ or λ a.e. x.
Hint: Use (b).

10. Let $[0,1]$ equipped with $\mathcal{M}$, the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Let m denote the Lebesgue measure on $[0,1]$ and let μ denote the counting measure on $[0,1]$. Show that $m \ll \mu$ but that there exists no f such that $dm = f d\mu$.