Math 623, Fall 2013: Problem set 5

1. Exercise 2, p. 90

2. Prove the following

   Theorem *Let $-\infty < a < b < \infty$ and let $f : \mathbf{R}^d \times [a,b] \to \mathbf{R}$ be such that $f(x,t)$ is integrable for any $t \in [a,b]$. Let*

   $$F(t) = \int f(x,t)\, dx\,.$$

   (a) Suppose that the map $t \mapsto f(x,\cdot)$ is continuous every x and that there exists an integrable function g such that $|f(x,t)| \le g(x)$ for all x, t. Then the function $F(t)$ is continuous.

   (b) Suppose that $\frac{\partial f}{\partial t}(x,t)$ exist and that there exists and integrable function h such that $|\frac{\partial f}{\partial t}(x,t)| \le h(x)$ for all x, t. Then the function $F(t)$ is differentiable and $F'(t) = \int \frac{\partial f}{\partial t}(x,t) dx$.

   Hint: Given t_0 let $\{t_n\}$ be an arbitrary sequence such that $\lim_n t_n = t_0$ and apply the Dominated Convergence Theorem.

3. **(a)** Consider the functions $f_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}$ defined on the interval $[0,1]$. Show that the sequence $\{f_n\}$ is not uniformly bounded, i.e., there exists no constant M such that $|f_n(x)| \le M$ for all $x \in [0,1]$ and all n.

   (b) Find a nonnegative function $g(x)$ such that $\int_{[0,1]} g(x)dx < \infty$ and $f_n(x) \le g(x)$ for all n and all $x \in [0,1]$. *Hint: Fix x and maximize $f_n(x)$ over n by replacing the discrete variable n by a continuous one.*

   (c) Compute $\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx$.

4. Compute $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\, dm$. Distinguish between $a > 0$, $a = 0$ and $a < 0$. Justify your computation carefully. *Hint:* Use the transformation of integrals under dilation.

5. (a) Suppose $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^\infty \int |f_n| dx < \infty$. Show that $\sum_{n=1}^\infty f_n(x)$ converges a.e. to an integrable function and $\int \sum_n f_n = \sum_n \int f_n$. *Hint:* Use DCT.

   (b) Prove that for $a > -1$ we have $\int_{[0,1]} x^a (1-x)^{-1} \log(x) dx = \sum_{n=1}^\infty \frac{1}{(a+n)^2}$

6. Consider the function $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$ and for $0 \le x < \infty$ Show that

(a) $\sum_{n=1}^{\infty} \int |f_n| dx = \infty$.

(b) $\sum_{n=1}^{\infty} \int f_n dx = 0$.

(c) $\sum_{n=1}^{\infty} f_n(x)$ is integrable on $[0, \infty)$ and $\int_{[0,\infty)} \sum_{n=1}^{\infty} f_n dx = \ln(b/a)$.