Math 623, Fall 2013: Problem set 4

1. If a function f is integrable then (see Proposition 1.12 in Chapter 2) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set A with $m(A) \leq \delta$ we have $\int_A |f|dx \leq \varepsilon$ (absolute continuity of the integral).

   We say that a sequence of functions $\{f_n\}_{n\geq 1}$ is **equi-integrable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set A with $m(A) \leq \delta$ we have $\int_A |f_n|dx \leq \varepsilon$ **for all** n. Prove the following

   Theorem *Let E is a set of finite measure, $m(E) < \infty$, and let $f_n : E \to \mathbf{R}$ be a sequence of functions which is equi-integrable. Show that if* $\lim_{n\to\infty} f_n(x) = f(x)$ *for almost every $x \in E$ then* $\lim_{n\to\infty} \int_E |f_n - f|dx = 0$.
   Hint: Use Egorov Theorem as in the bounded convergence theorem.

2. Exercise 6, p. 91

3. Exercise 8, p. 91

4. Exercise 10, p.91

5. Exercise 11, p. 91

6. In class proved first the bounded convergence theorem (using Egorov Theorem). We then proved Fatou's Lemma (using the Bounded Convergence theorem) and deduced from it the Monotone Convergence Theorem. Finally we proved the Dominated Convergence Theorem (using both the monotone Convergence Theorem and the Bounded Convergence Theorem).

   There are other ways to prove this sequence of results. For example

   (a) Deduce Fatou's Lemma from the Monotone Convergence Theorem by showing that for any sequence of measurable functions $\{f_n\}$ we have

   $$\int \liminf_n f_n \, dm \leq \liminf \int f_n \, dm$$

   Hint: Note that $\inf_{n\geq k} f_n \leq f_j$ for any $j \geq k$ and thus $\int \inf_{n\geq k} f_n \, dm \leq \inf_{j\geq k} \int f_j \, dm$.

   (b) Deduce the dominated Convergence Theorem from Fatou's Lemma. *Hint:* Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.