Math 623, Fall 2013: Problem set 3

1. Let $I = [a, b]$ be a finite closed interval. The function f is said to be Lipschitz continuous on I if there exists a constant L such that for all $x, y \in I$ we have

$$|f(x) - f(y)| \leq L|x - y| .$$

   (a) Show that if f is Lipschitz continuous then f is continuous.

   (b) Show that if f is continuously differentiable on I then f is Lipschitz continuous.

   (c) Show that if $A \subset I$ has measure 0 and f is Lipschitz continuous then $f(A)$ has measure 0. *Hint:* Use the definition of the exterior measure.

2. Prove that if f is measurable and $f = g$ almost everywhere then g is measurable.

3. Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is finite-valued. Show that f is measurable if and only if $f^{-1}(A)$ is measurable for every Borel set A.

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable. Show that f and f' are measurable functions.

5. (a) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a monotone function. Show that $f^{-1}(A)$ is a Borel set for every Borel set A. In particular f is measurable.

   (b) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a one to one continuous function. Show that f maps Borel sets onto Borel sets.

6. (a) Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $f(A)$ is not measurable.

   (b) Give an example of a function $g : \mathbf{R} \to \mathbf{R}$ and a measurable set A such that $g^{-1}(A)$ is not measurable.

   (c) Give an example of a measurable set such which is not a Borel set.

   (d) Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable.

   Hint: Let $F : [0, 1] \to [0, 1]$ be the Cantor Lebesgue function constructed in Exercise 2, chapter 1, and extend it to $\mathbf{R}$ by setting $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$. Finally let

$$f(x) = x + F(x) .$$

   Use problem 6(b) to show that if C is the middle third cantor set then $m(f(C)) = 1$ and so f maps a set of measure 0 onto a set of positive measure.

   Using this function f, the problem 5 in Problem set 2 (Exercise 32 (b) in the book), and problem 6 again, you can now deduce (a), (b), (c), (d).