Math 623, Fall 2013: Problem set 2

1. Show that a countable union of set of exterior measure 0 has exterior measure 0 using directly the definition of exterior measure. In particular any countable set has measure 0, e.g. the rational numbers in $[0, 1]$.

2. Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbf{R}^n$. We define

$$\begin{aligned} \limsup_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for infinitely many } n\right\} \\ \liminf_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for all but finitely many } n\right\}\,. \end{aligned}$$

   (a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\,, \quad \text{and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\,.$$

   (b) Show that

$$\begin{aligned} m\left(\liminf_{n\to\infty} E_n\right) &\leq \liminf_{n\to\infty} m\left(E_n\right) \\ \limsup_{n\to\infty} m\left(E_n\right) &\leq m\left(\limsup_{n\to\infty} E_n\right) \quad \text{provided } m\Big(\bigcup_{n=1}^{\infty} E_j\Big) < \infty\,. \end{aligned} \tag{1}$$

   (c) Exercise 16, p.42. (Borel-Cantelli Lemma).

3. Suppose that A is a measurable set in $\mathbf{R}^d$ with $m(A) > 0$. Show that for any $q < m(A)$ there exist a measurable set $B \subset A$ with $m(B) = q$. *Hint:* Prove it first for the case that $m(A) = p < \infty$. Use then the intermediate value theorem for $A \cap B_R(0)$.

4. Exercise 28, p.43

5. Exercise 32, p.43

6. Exercise 33, p.43