Math 623: Homework 5

1. Exercise 21, p. 94

2. Let X be a Banach space X equipped with a product $x \times y$. If X is an algebra and if

$$\|x \times y\| \leq \|x\|\|y\|\,, \qquad \text{for all } x, y \in X$$

then X is called a Banach algebra.

   (a) Show that $L^1(\mathbf{R}^d)$ equipped with the convolution product $f \star g$ is a complex Banach algebra.

   (b) Show that $L^1(\mathbf{R}^d)$ equipped with pointwise multiplication fg is not an algebra.

3. Exercise 23, p.94 (This shows that the Banach algebra $L^1(\mathbf{R}^d)$ has no unit.)

4. Exercise 24, p. 95

5. Exercise 25, p.95

6. Consider the following two functions defined on $\mathbf{R}$.

$$\begin{aligned} h^{(1)}(\xi) &= c_1 e^{-\delta 2\pi|\xi|}\,. \\ h^{(2)}(\xi) &= c_2(1 - \delta 2\pi|\xi|)\chi_{[-\frac{1}{2\pi\delta}\,,\,\frac{1}{2\pi\delta}]} \end{aligned}$$

Compute their Fourier transforms $K_\delta^{(1)}(y) = \hat{h}_1(y)$ and $K_\delta^{(2)}(y) = \hat{h}_2(y)$ and show that for suitable choices of c_1 and c_2 (which you should determine) we have

$$\begin{aligned} &(i) && \int K_\delta^{(j)}(y)dm(y) = 1 \\ &(ii) && \lim_{\delta \to 0} \int_{y \geq |\eta|} K_\delta^{(j)}(y)dm(y) = 0\,, \text{ for any } \eta > 0\,. \end{aligned} \tag{1}$$

i.e. $K_\delta^{(j)}$ are **good kernels**. ($K(1)_\delta$ is known as *Abel's kernel* while $K^{(2)}$ is *Fejer's kernel*

Remark: In proving the Fourier inversion formula we have already encountered the good Kernel $K_\delta(y) = \hat{g}(y) = \delta^{-d/2}e^{-\pi|y|^2/\delta}$ (*Gauss Kernel*) which is the Fourier transform of $g(\xi) = e^{-\pi\delta|\xi|^2}$.

7. Show that if $f \in L^1(\mathbf{R}^d)$ then its Fourier transform $\hat{f}(\xi)$ is uniformly continuous.

8. The Riemman-Lebesgue Lemma states that if $f \in L^1$ then $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$. Prove the statement first for a characteristic functions of rectangle, i.e. $f = \chi_R$ and deduce from this that the statement holds for all $f \in L^1$.

9. Problem 1, p. 95

10. Exercise 4, p. 146

11. Exercise 5, p.146

12. Exercise 7, p. 147