# Math 623: Homework 4

1. Exercise 4, p.90
2. Exercise 17, p. 93
3. Exercise 18, p.93
4. Exercise 19, p.93
5. Compute and justify your computations
   (a) $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dm$.
   (b) $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\, dm$.
   (c) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\, dm$. Distinguish between $a > 0$, $a = 0$ and $a < 0$.
6. Consider the function $f(x, y) = ye^{-(1+x^2)y^2}$ if $x \geq 0$ and $y \geq 0$ and 0 otherwise. Integrate this function over $\mathbf{R} \times \mathbf{R}$ to show that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$
7. Investigate the existence and equality of the integral $\int_{[0,1]\times[0,1]} f(x, y)dm(x, y)$, and the iterated integrals $\int_{[0,1]}(\int_{[0,1]} f(x, y)dm(x))dm(y)$, and $\int_{[0,1]}(\int_{[0,1]} f(x, y)dm(y))dm(x)$ for the following functions
   (a) $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$
   (b) $f(x, y) = (x - \frac{1}{2})^{-3}$ if $0 < y < |x - \frac{1}{2}|$, $f(x, y) = 0$ otherwise.
8. Let $E$ be subset of $\mathbf{R}^d$ with finite measure. For any two functions $f$, $g$ measurable on $E$ let $\rho_E(f, g) = \int_E \frac{|f-g|}{1+|f-g|}dm$.
   (a) Show that $\rho_E(f, g)$ defines a metric on the set of measurable functions defined on $E$.
   (b) Let $\{f_n\}$ be a sequence of measurable functions defined on $E$ . Show that $\lim_{n\to\infty} \rho_E(f_n, g) = 0$ if and only if $f_n$ converges to $f$ in measure.
   (c) Show that the assumption that $E$ has finite measure is necessary. *Hint:* Consider $f_n(x) = nx^{-1}$.
9. Suppose that $f_n$ converge to $f$ in measure and that there exists a function $g \in L^1$ such that $|f_n| \leq g$ a.e. for all $n$. Show that $f$ is integrable and $\lim_{n\to\infty} \int |f - f_n|dm = 0$.
10. The following result about sequences is often very useful. Let $\{x_n\}$ be a sequence of real numbers. Show that $\lim_n x_n = x$ if and only if every subsequence of $\{x_n\}$ has a subsequence which converges to $x$.
11. Let $\{f_n\}$ be a sequence of nonnegative functions such that $f_n \to f$ a.e and $\int f_n dm \to \int f dm$. Show that for any measurable set $E$,

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm\,.$$

   *Hint:* Apply Fatou's Lemma to $f\chi_E$ and $f\chi_{E^c}$ and use the previous problem.