Math 623: Homework 3

1. In class we have first the bounded convergence theorem (using Egorov Theorem). We then proved Fatou's Lemma (using the Bounded Convergence theorem) and deduced from it the Monotone Convergence Theorem. Finally we prove the Dominated Convergence Theorem (using both the monotone Convergence Theorem and the Bounded Convergence Theorem).

   There are other ways to prove this sequence of results.

   (a) Deduce Fatou's Lemma from the Monotone Convergence Theorem by showing that for any sequence of measurable functions $\{f_n\}$ we have

   $$\int \liminf_n f_n \, dm \le \liminf \int f_n \, dm$$

   Hint: Note that $\inf_{n \ge k} f_n \le f_j$ for any $j \ge k$ and thus $\int \inf_{n \ge k} f_n \, dm \le \inf_{j \ge k} \int f_j \, dm$.

   (b) Deduce the dominated Convergence Theorem from Fatou's Lemma. *Hint:* Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

2. Prove the following Theorem which deals with exchanging limit and differentiation with integrals.

   Theorem 0.0.1 *Let $-\infty < a < b < \infty$ and let $f : \mathbf{R}^d \times [a, b] \to \mathbf{R}$ be such that $f(x, t)$ is integrable for any $t \in [a, b]$. Let*

   $$F(t) = \int f(x, t) \, dm \, .$$

   (a) Suppose that $f(x, \cdot)$ is continuous in t for every x and that there exists an integrable function g such that $|f(x, t)| \le g(x)$ for all x, t. Then the function $F(t)$ is continuous.

   (b) Suppose that $\frac{\partial f}{\partial t}$ exist and that there exists and integrable function h such that $|\frac{\partial f}{\partial t}(x, t)| \le h(x)$ for all x, t. Then the function $F(t)$ is differentiable.

   Hint: Given t_0 let $\{t_n\}$ be an arbitrary sequence such that $\lim_n t_n = t_0$ and apply the Dominated Convergence Theorem.

3. Use the previous exercise and the identity $\int e^{-tx^2} \, dm = \sqrt{\pi/t}$ to show that $\int x^{2n} e^{-x^2} \, dm = \frac{(2n)!\sqrt{\pi}}{4^n n!}$

4. Exercise 6, p. 91

5. Exercise 7, p. 91

6. Exercise 8, p. 91

7. Exercise 9, p. 91

8. Exercise 10, p. 91

9. Exercise 11, p. 91

10. Exercise 15, p.92