Math 623: Homework 1

1. Exercise 1, p. 37
2. Exercise 2, p. 37
3. Exercise 3, p. 38
4. Exercise 4, p. 38
5. Exercise 9, p. 40
6. Show that a countable union of set of measure 0 has measure 0 directly from the definition

$$m(E) = 0 \quad \text{if} \quad 0 \;=\; \inf\{\sum_{j=1}^{\infty} |Q_j| \,;\, Q_j \text{ closed cubes}\,, E \subset \bigcup_{j=1}^{\infty} Q_j\}\,.$$

   In particular any countable set has measure 0, e.g. the rational numbers in $[0,1]$. What is the Lebesgue measure of the set of irrational numbers in $[0,1]$?

7. Problem 4, p.47: As a background for this problem we recall the definition of the Riemann integral of a function. Let $f : [a,b] \to \mathbf{R}$ be a *bounded* function. Let P be a finite partition of $[a,b]$, i.e., $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. We define the lower and upper Darboux sums, $L(f,P)$ and $U(f,P)$ by

$$U(f,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup_{t_{i-1} \le x \le t_i} f(x)\,, \quad L(f,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \inf_{t_{i-1} \le x \le t_i} f(x)\,.$$

   A bounded function f is called Riemann integrable if

$$\sup_P L(f,P) = \inf_P U(f,P)$$

   where the inf and sup are taken over all finite partition P. In other words, for any $\epsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) \le \epsilon$. If f is Riemann integrable then the Riemann integral of f, $\int_a^b f(x)dx$ is defined to be $\inf U(f,P)$ (or $\sup L(f,P)$).

8. Construct a sequence of functions $f_n : [0,1] \to \mathbf{R}$ such that

   (i) For all x the sequence $\{f_n(x)\}$ is decreasing to $f(x)$.

   (ii) f_n is Riemann integrable and $\int_0^1 f_n dx = 1$ for all n.

   (iii) f is not Riemann integrable.

   To do this consider an enumeration $r_1, r_2, \cdots$ of the rationals in $[0,1]$ and use exercises 6 and 7. The function f_n do not need to be continuous.

9. Exercise 10, p.40. Note that this is a stronger version than the problem 9: the f_n are now continuous(!).

10. Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbf{R}^n$. We define

$$\begin{aligned}\limsup_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for infinitely many } n\right\}\\ \liminf_{n\to\infty} E_n &= \left\{x \in \mathbf{R}^d\,;\, x \in E_n \text{ for all but finitely many } n\right\}.\end{aligned}$$

(a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\,, \qquad \text{and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\,.$$

(b) Show that

$$\begin{aligned} m\left(\liminf_{n\to\infty} E_n\right) &\leq \liminf_{n\to\infty} m\left(E_n\right)\\ \limsup_{n\to\infty} m\left(E_n\right) &\leq m\left(\limsup_{n\to\infty} E_n\right) \qquad \text{provided } m\Big(\bigcup_{n=1}^{\infty} E_j\Big) < \infty\,. \end{aligned} \tag{1}$$

(c) Problem 16, p.42. (Borel-Cantelli Lemma).

11. Problem 1, p.46