Math 523H–Homework 9

1. (a) Use a geometric series to write down a series for $\frac{1}{1+x^2}$.

   (b) Use your result in (a) to write down a series expansion for $\arctan(x)$ for $-1 < x < 1$. Justify carefully all the steps.

2. Consider the sequences of functions on $[0, 1]$:

$$(a)\ f_n(x) = \frac{nx}{(1+n^2x^2)^2}\,, \quad (b)\ f_n(x) = \frac{n^2x}{(1+n^2x^2)^2}$$

   Compute the limits $\lim_{n\to\infty} f_n(x)$. Determine if the convergence is uniform (compute the maximum of f_n). Finally determine whether

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx\,.$$

   Hint: Use some calculus here.

3. Find a series for the function $g(t) = \int_0^x e^{-t^2}dt$ by using the series for the exponential function. Justify carefully all steps.

4. Suppose that $f_n(x)$ converges uniformly on $[a, b]$. Show that $F_n(x) = \int_a^x f_n(t)dt$ converges uniformly on $[a, b]$.

5. Show that if $g(x)$ is differentiable at x_0 and $g(x_0) \neq 0$ then $1/g(x)$ is differentiable at x_0 and compute the derivative of $1/g(x)$. Prove this in two ways:

   (a) Using the definition of the derivative as a limit.

   (b) Using our third formulation of the derivative (Caratheodory formulation) (g is differentiable if there exists a function $\phi(x)$ continuous at x_0 such that $g(x) = g(x_0) + \phi(x)(x - x_0)$).

6. Using one of the equivalent definitions of the derivative show that $f(x) = \sqrt{x}$ is differentiable for $x > 0$ and that $g(x) = x^{1/3}$ is differentiable at $x \neq 0$. Show also that $x^{1/3}$ is not differentiable at $x = 0$.

7. Consider the function $f_n = x^n \sin(1/x^3)$ for $x \neq 0$ and $f(0) = 0$ where n is a non-negative integer. For which values of n is $f(x)$ (a) continuous? (b) differentiable? (c) twice differentiable? When are the derivatives continuous? *Hint: You may use that* $\sin(x)$ *and* $\cos(x)$ *are differentiable and their derivatives without proving it.*

8. The function $\sin(x)$ is bijective on $[-\pi/2, \pi/2]$ and $\arcsin(x)$ is its inverse function. Compute the derivative of *arcsin*(x). *Hint: You may use the derivative of* $\sin(x)$ *without proving it.*