Math 523H–Homework 7

1. Show that $f(x) = x^2$ is integrable on $[0, 2]$ and compute $\int_0^2 f(x)dx$ using the definition with upper and lower Darboux sum. *Hint:* What is $1 + 2^2 + 3^2 + \cdots + n^2$?

2. In numerical analysis to compute the integral $\int_a^b f(x)dx$ one divide the interval into N subinterval of equal length $h = (b - a)/N$ that is we have $x_i = a + i(b - a)/N$ and use approximations. For example the trapezoidal rule is

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{h}{2} \left(f(x_i) + f(x_{i+1}) \right). \tag{1}$$

It means that one approximates the area under the graph between x_{i-1} and x_i be the trapezoidal area $\frac{h}{2}(f(x_i) + f(x_{i+1})$. Prove that as $N \to \infty$ the trapezoidal rule converge to the value of $\int_a^b f(x)dx$ by rewriting it as a suitable Riemann sum.

3. (a) Suppose $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is integrable. Show that f is integrable on $[a, b]$ and $[b, c]$ and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

   (b) A function $f : [a, b] \to \mathbb{R}$ is called piecewise continuous if there exists a division $D = \{x_0, x_1, \cdots, x_n\}$ such that $f : (x_{i-1}, x_i) \to \mathbb{R}$ is uniformly continuous. Show that if f is piecewise continuous then f is integrable.

4. (a) Prove that if f is integrable on $[a, b]$ then $|f|$ is integrable on $[a, b]$.

   (b) Prove that if $f(x)$ and $g(x)$ are integrable on $[a, b]$ then $h(x) = \max\{f(x), g(x)\}$ is integrable on $[a, b]$

5. Suppose that $f(x) = 1$ if $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ and $f(x) = x$ otherwise. Prove that $f(x)$ is integrable on $[0, 1]$. *Hint:* Consider the intervals $[0, \epsilon]$ and $[\epsilon, 1]$ separately.

6. Show that the function $f(x) = \sin(\frac{1}{x})$ is integrable on $[0, 1]$.

7. Suppose $f : [a, b] \to \mathbb{R}$ is (a) continuous, (b) non-negative, that is $f(x_0) \geq 0$ for all x, and (c) there exists x_0 such that $f(x_0) > 0$. Show that $\int_a^b f(x)dx > 0$. Show that all three assumptions are necessary.

8. Suppose f and g are two continuous function on $[a, b]$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that that there exists an $x \in [a, b]$ such that $f(x) = g(x)$.