Math 523H–Homework 7

1. Show by an explicit ϵ-δ argument that $f(x) = x^2$ is uniformly continuous on $[0, 3]$.

2. Show that if f is uniformly continuous on a bounded set A then f is bounded on A. *Hint*: Do a proof by contradiction and use the fact proved in class that a uniformly continuous function maps Cauchy sequences into Cauchy sequences, and Bolzano-Weierstrass Theorem as well.

3. For the following functions determine if they are uniformly continuous on the given set. Justify your answer by using appropriate theorems or proof.
   (a) $f(x) = x^3 + \sin(x)$ on $[0, 2]$. (b) $f(x) = \frac{x^2+x-6}{x-2} + \cos(2x)$ on $(0, 2)$.
   (c) $f(x) = \frac{1}{1-x}$ on $[0, 1)$. (d) $f(x) = x^2 \sin(\frac{1}{x})$ on $(0, 1]$.
   (e) $\cos(2x)$ on $(-\infty, \infty)$, (f) e^x on $(-\infty, \infty)$.

4. Show that the function $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to 0 on $[0, 1]$. *Hint:* For each n compute the maximum and minimum of f_n on the interval $[0, 1]$.

5. Consider the function $f_n(x) = \frac{x^n}{1+x^n}$. Does f_n converge uniformly on $[0, \infty)$?

6. Consider the function given $f(x) = \sum_{k=1}^{\infty} \frac{x^n \cos(2nx)}{n^2 2^n}$ on the interval $[-2, 2]$. Show that that the function f is continuous.

7. Consider the sequence of functions $f_n(x) = (n + 1)x^n(1 - x)$ on the interval $[0, 1]$.
   (a) Compute $f(x) = \lim_{n\to\infty} f_n(x)$. Is f continuous?
   (b) Show that f_n does not converge uniformly to f.
   Hint: Find the maximum of $f_n(x)$.

8. Consider the series
$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{1+x^2} \left(\frac{1}{1+x^2} \right)^n .$$
   (a) Show that the series converges absolutely for all $x \in \mathbb{R}$.
   (b) Show that the series does not converges uniformly on $[-1, 1]$.
   (c) Compute $f(x)$. Is is continuous?