Math 523H–Homework 6

1. Use an $\epsilon - \delta$ argument to show that the following functions are continuous:
   (a) $f(x) = \sqrt{x}$ for any point $x_0 \geq 0$,
   (b) $f(x) = x^3$ for any point x_0. *Hint:* $(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$.
   (c) $f(x) = x^3 \sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(0) = 0$ at the point $x = 0$.

2. Consider the function defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

   Show by his function is not continuous at 0 with (a) a $\epsilon - \delta$ argument and (b) a sequence argument.

3. (a) Let f and g be continuous functions on $[a, b]$ and assume that $f(a) \leq g(a)$ and $g(b) \leq f(b)$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.
   (b) Show that the equation $x = \cos(x)$ has at least one solution in $(0, \pi/2)$.

4. Prove that a polynomial of odd degree has at least one real root.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous which satisfies $f(a)f(b) < 0$ for some $a \neq b$. Show that there exists and x such that $f(x) = 0$.

6. (a) Consider the function $f(x) = \frac{x^3 - 3x^2 - 13x + 15}{x^2 - 1}$. At which point is this function continuous? Describe the discontinuities of the function.
   (b) Consider the function $f(x) = \frac{\sqrt{1 + 3x^2 + 2x^4} - 1}{x^2}$ for $x \neq 0$. Can you extend f to a continuous function on $\mathbb{R}$?

7. (a) Suppose that $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Show that f is discontinuous at every x.
   (b) Suppose that $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational. Show that f is continous at 0 but discontinuous at every other point.

8. (a) If f is a continuous function such that $f(x) = 0$ for every rational x, show that $f = 0$.
   (b) If f and g are two continuous functions such $f(x) = g(x)$ for every rational x, show that $f = g$.
   (c) Suppose that f is continuous function which satisfies $f(x + y) = f(x) + f(y)$ for all x and y. Show that $f(x) = ax$ for some constant a. *Hint:* Consider first $x = n$ an integer, then $x = \frac{1}{n}$, then x rational.