Math 523H–Homework 5

1. Consider the series $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$. Give at least four different proofs that the series converges.

2. Suppose that $\{a_n\}$ is a sequence such that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Suppose that c_n is a bounded sequence. Show that $\sum_{n=1} a_n c_n$ converges absolutely.

3. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ converges with $c \neq 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} b_n$ converges absolutely.

4. Determine which of the following series converge, and which ones converges absolutely. Justify your answer by stating the appropriate criterion (root test, ratio test, alternating series tests, comparison test.....)

   (a) $\sum \frac{n^4}{2^n}$.

   (b) $\sum \frac{100^n}{\sqrt{n!}}$

   (c) $\sum \frac{\cos^2(n^2)}{n^2}$

   (d) $\sum \frac{(-1)^n}{n^{1/3}}$

5. Determine which of the following series converge, and which ones converges absolutely or not. Justify your answer by stating the appropriate criterion (root test, ratio test, alternating series tests, comparison test.....)

   (a) $\sum \frac{n^2 3^n}{100^n - n}$.

   (b) $\sum \frac{2^{3n+2}}{n!}$.

   (c) $\sum \frac{\cos(\frac{n\pi}{2})}{\sqrt{n}}$.

   (d) $\sum (-1)^n \frac{n+1}{n^3+1}$

6. In class we prove the convergence of $\sum_n \frac{1}{n^2}$ by comparing to the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ .... Proceeding in the same way show that the series $\sum_n \frac{1}{n^{3/2}}$ converges by comparing it to the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ ...

Hint: Work on the expression $\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n}}$.

7. In general it is hard to compute the actual values of an infinite series. Here are some examples where you can compute it.

   (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

   (b) Prove that $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint :* Use that $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

   (c) Use part (b) to compute $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

8. (a) Show that the series

$$\begin{aligned} \sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5! - \cdots} \\ \cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{2!} - \cdots \end{aligned}$$

   converge absolutely for any value of x and y.

   (b) Using the Cauchy product of two series

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n \quad \text{with} \quad c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

   which is justified if the series converges absolutely show the formula

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$