Math 523H–Homework 4

1. For each of the following sequences, compute $\sup\{s_n\}$, $\inf\{s_n\}$, $\limsup\{s_n\}$ and $\liminf\{s_n\}$ and determine all the accumulation points.

   (a) $s_n = 7^{(-\frac{1}{2})^n}$

   (b) $s_n = 3^{(-1)^n} + \sin(\frac{n\pi}{2})$

   (c) $s_n = (-1)^n \frac{n+5}{n}$

   (d) $s_n = n\cos(\frac{n\pi}{4})$

2. Construct a sequence whose accumulation points are all the non-negative integers.

3. Consider the following sequences

$$\{s_n\} = \{0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \cdots\}$$

$$\{t_n\} = \{2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \cdots\}$$

   Compute (a) $\liminf s_n + \liminf t_n$, (b) $\liminf(s_n + t_n)$ (c) $\liminf s_n + \limsup t_n$, (d) $\limsup(s_n + t_n)$, (e) $\limsup s_n + \limsup t_n$, (f) $\liminf(s_n t_n)$ (g) $\limsup(s_n t_n)$.

4. Show the following facts:

   (a) If the sequence $\{s_n\}$ converges then every subsequence of $\{s_n\}$ converges to the same limit.

   (b) A sequence $\{s_n\}$ converges if and only if $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

5. **Three equivalent definitions of** $\limsup$**:** Suppose $\{s_n\}$ is a bounded sequence. In class we have defined $\limsup s_n$ as

$$\xi = \limsup_{n\to\infty} s_n = \sup\{x \mid s_n > x \text{ for infinitely many } n\}$$

   and have established in the Bolzano-Weierstrass theorem that

$$\xi = \limsup_{n\to\infty} s_n \text{ is the largest accumulation point of the sequence } \{s_n\}$$

   which gives another characterization of $\limsup$. Here is a third one: prove the formula

$$\xi = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup_{k\geq n}\{s_k\}\,.$$

   Hint: Look at your class notes.

6. Write down the three equivalent definitions of lim inf similarly to Problem 4. (You do not need to prove it.) Show also that

$$\liminf s_n = -\limsup(-s_n)$$

and

$$\liminf(s_n + v_n) \geq \liminf(s_n) + \liminf(v_n)$$

7. Prove that if $\{s_n\}$ and $\{t_n\}$ are bounded sequences of non-negative numbers then

$$\limsup_n(s_n t_n) \leq \limsup(t_n) \limsup(s_n)\,.$$

What happens if you relax the condition that s_n and t_n are non-negative?

8. Show that every sequence $\{s_n\}$ has a subsequence which is monotone (either decreasing or increasing).
Hint: Call a term s_n dominant if $s_n > s_m$ for all $m > n$. Show that if there are infinitely many dominant terms there is a monotone decreasing subsequence and if there is finitely many dominant terms there is a monotone increasing subsequence.