Math 523H–Homework 3

1. Show that the relation $\sim$ between Cauchy sequences defined as $\{s_n\} \sim \{t_n\}$ if $\lim(s_n - v_n) = 0$ is an equivalence relation that is it is reflexive, symmetric and transitive.

2. Consider two real numbers s and t, that two equivalence classes of Cauchy sequences of rational numbers $s = \overline{\{s_n\}}$ and $t = \overline{\{t_n\}}$. In this problem we study the product of real numbers.

   (a) Given two Cauchy sequence $\{s_n\}$ and $\{t_n\}$ show that the sequence $\{s_n \cdot t_n\}$ is a Cauchy sequence.

   (b) Show that if $\{s_n\} \sim \{s'_n\}$ and $\{t_n\} \sim \{t'_n\}$ show that $\{s_n \cdot t_n\} \sim \{s'_n \cdot t'_n\}$.

   This allows us to define the product $s \cdot v$ as $s \cdot v = \overline{\{s_n \cdot t_n\}}$.

3. For your peace of mind verify carefully that if we define real numbers as equivalence classes of Cauchy sequences then real numbers satisfy the distributive law: given $s = \overline{\{s_n\}}$, $t = \overline{\{t_n\}}$, and $v = \overline{\{v_n\}}$ then we have $s(t + v) = st + sv$.

4. Suppose $\{a_n\}$ is a convergent sequence with $\lim_{n\to\infty} a_n = a$ and define a new sequence b_n by

$$b_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(a_1 + \cdots a_n)$$

   that is b_n is the average of the first n terms of the sequence a_n. Show that $\{b_n\}$ is a convergent sequence and $\lim_{n\to\infty} b_n = a$.

5. The Fibonacci sequence is defined by $F_0 = 1, F_1 = 0$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$. We find $\{F_n\} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots\}$. We are going to show that the ratio F_n/F_{n-1} converges to the golden ratio $\phi = (1 + \sqrt{5})/2$.

   (a) Show that the ratio $s_n = \dfrac{F_n}{F_{n-1}}$ satisfies the recursion relation $s_n = 1 + \dfrac{1}{s_{n-1}}$

   (b) Show that if s_n converges to $s \neq 0$ then s_n must converges to ϕ.

   (c) Pick some $\delta > 0$ (not too big, for example $\delta = 1/10$ will do). Show by induction that (for $n \geq 2$) we have $s_n \geq 1 + \delta$

   (d) Using part (b) and (c) show that $|s_{n+1} - s_n| \leq \frac{1}{(1+\delta)^2}|s_n - s_{n-1}|$ and deduce from this that $|s_{n+1} - s_n| \leq \frac{1}{(1+\delta)^{2n}}$.

   (e) Use problem 7 in Hwk 2 to wrap things up and conclude.