Math 523H–Homework 2

1. Let $\{t_n\}$ be a bounded sequence and $\{s_n\}$ be a convergent sequence with $\lim_{n\to\infty} s_n = 0$. Show that $\lim_{n\to\infty}(s_n t_n) = 0$.

2. Give a detailed proof of the following

   Theorem *For any sequence $\{s_n\}$ with $s_n > 0$ we have $\lim_{n\to\infty} s_n = +\infty$ if and only if $\lim_{n\to\infty} \frac{1}{s_n} = 0$.*

3. (a) As proved in class if $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = t > 0$ then $\lim_{n\to\infty}(s_n t_n) = +\infty$. Construct examples of sequences $\{s_n\}$ and $\{t_n\}$ with $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = 0$ and

      i. $\lim_{n\to\infty}(s_n t_n) = +\infty$

      ii. $\lim_{n\to\infty}(s_n t_n) = c$ for any arbitrary constant c.

      iii. The sequence $s_n t_n$ is bounded but not convergent.

   (b) Suppose that $\lim_{n\to\infty} s_n = +\infty$ and t_n is a bounded sequence. Show that $\lim_{n\to\infty}(s_n + t_n) = +\infty$.

4. Suppose $\{s_n\}$ is a sequence such that $\lim\limits_{n\to\infty} \left|\dfrac{s_{n+1}}{s_n}\right| = L$ exists.

   (a) Show that if $L < 1$ then $\lim_{n\to\infty} s_n = 0$.
   Hint: Pick b such that $L < b < 1$ and choose N so large that $\left|\frac{s_{n+1}}{s_n}\right| < b < 1$ for all $n \geq N$. Then show that for $n \geq N$ we have $|s_n| \leq b^{n-N}|s_N|$.

   (b) Show that if $L > 1$ then $\lim_{n\to\infty} |s_n| = \infty$.
   Hint: Proceed as in (a) or use Problem 2.

5. Use Problem 4 to prove that

   (a) Let $p > 0$. Show that

   $$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \leq 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exists} & \text{if } a < -1 \end{cases} .$$

   (b) Show that for any number a we have $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0$.

6. Use Problems 3 and 5 to explain why

   (a) $\lim\limits_{n\to\infty} \dfrac{n^4 + n}{n^2 - 5} = +\infty$.

(b) $\lim_{n\to\infty} \frac{2^n}{n^{10}+1} + (-5)^n = +\infty$

(c) $\lim_{n\to\infty} \frac{10^n}{n!} - \frac{7^n}{n^3} = -\infty$

7. (a) Suppose $\{s_n\}$ is a sequence such that for all n we have

$$|s_n - s_{n+1}| \leq \alpha^n$$

for some $\alpha < 1$. Then prove that $\{s_n\}$ is a Cauchy sequence.

Hint: Use the geometric series $1 + \alpha + \alpha^{k-1} = \frac{1-\alpha^k}{1-\alpha}$ to bound $|s_n - s_{n+k}|$.

(b) Suppose we are given a decimal expansion $k.d_1d_2d_3\cdots$ of a real number where k is an integer and $d_i \in \{0, 1, 2, \cdots, 9\}$. Use part (a) to show that

$$s_n = k + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}$$

is a Cauchy sequence.

8. Consider the sequence

$$s_n = \frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \frac{1}{7\cdot 11} + \cdots + \frac{1}{(2n-1)(2n+3)}$$

Show that $\{s_n\}$ is a Cauchy sequence and compute its limit.

Hint: Compute the partial fraction expansion of the $\frac{1}{(2j-1)(2j+3)}$.