Math 523H–Homework 11

1. Compute the convergence radius of the following series

   (a) $\sum_{n=0}^{\infty} n! x^n$

   (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

   (c) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} x^n$

   (d) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

   (e) $\sum_{n=0}^{\infty} 2^{-n} x^{3n}$ *Hint:* The answer is not 2.

   (f) $\sum_{n=0}^{\infty} x^{n!}$

   (g) $\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} x^n$

2. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Justifying all your steps show that $f'(x) = f(x)$. *Hint:* You are not allowed to use that $f(x) = e^x$ and calculus.

3. Let $c_n = \left(\frac{4+2(-1)^n}{5}\right)^n$.

   (a) Compute $\limsup_n |c_n|^{1/n}$, $\liminf_n |c_n|^{1/n}$, $\limsup_n \left|\frac{c_{n+1}}{c_n}\right|$, $\liminf_n \left|\frac{c_{n+1}}{c_n}\right|$

   (b) Find the radius of convergence for the series $\sum_n c_n x^n$.

   (c) Determine the exact interval in which the series $\sum_n c_n x^n$ converge.

4. Using your knowledge of derivatives of $\sin(x)$ and $\cos(x)$, use Taylor theorem to derive the power series for $\sin(x)$. (You have to bound the remainder term!). What is the convergence radius?

5. Consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{n!} = \frac{\cos(2x)}{1!} + \frac{\cos(4x)}{2!} + \frac{\cos(8x)}{3!}$$

   (a) Show that the series and all its derivatives converges uniformly on $\mathbb{R}$.

   (b) Compute the Taylor series of the function as the origin and shows that it diverges at all $x \neq 0$.

6. Show that $\int_2^{\infty} \frac{1}{x(\ln x)^k} dx$ converges for $k > 1$ and diverges for $k \leq 1$

7. Determine if the following integrals are convergent or divergent. If the integrant depends on k then your answer will spend on k as well.

$$(a) \int_0^{\infty} \frac{x}{1+x^3} dx\,, \quad (b) \int_0^{\infty} \frac{1}{\sqrt{1+x^3}} dx \quad (c) \int_1^{\infty} \frac{x^k}{\sqrt{x^2-1}} dx \quad (d) \int_0^{\infty} x^k e^{-x^2} dx$$

   Hint: Change of variable might be useful

8. Show that the integral $\int_0^\infty \sin(x)dx$ diverges but that the Fresnel integral $\int_0^\infty \sin(x^2)dx$ converges. *Hint:* Set $u = x^2$ and then integrate by parts.