Math 523H–Homework 10

1. Assume that $f : (a,b) \to \mathbf{R}$ is twice continuously differentiable on (a,b), that is $f'(x)$ and $f''(x)$ exists and are continuous for all $x \in (a,b)$. Show that for $x \in (a,b)$ we have
$$\lim_{h\to 0} \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} = f'(x),$$
and
$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$
Hint: Use the mean value theorem (Lagrange Theorem).

2. Show that if f is differentiable on (a,b) and $f'(x)$ is bounded on (a,b) then f is bounded on (a,b).

3. Let $f : [a,b] \to \mathbb{R}$ be continuous and n times differentiable on (a,b). Show that if $f(x)$ has $n+1$ zeros in $[a,b]$ then there exists $\xi \in (a,b)$ such that $f^{(n)}(\xi) = 0$. *Hint:* Apply Rolle's theorem several times.

4. Show that the following functions are differentiable and compute their derivatives.
$$(a)\, f(x) = \int_0^{\sin(x)} \sqrt{1+t^2}dt\,, \quad (b)\, g(x) = \int_{x^2}^{x} e^{-t^2}\, dt\,.$$

5. **(Logarithms and exponentials).** Let $\ln(x)$ be the function defined for $x > 0$ by
$$\ln(x) = \int_1^x \frac{1}{t} dt$$
Invoking appropriate theorems from class (you will need quite a few of them) show the following
   (a) Show that $\ln(x)$ is a strictly increasing differentiable function with derivative $\ln'(x) = 1/x$.
   (b) Show that $\ln(ab) = \ln(a) + \ln(b)$, $\ln(1/a) = -\ln(a)$ and $\ln(a^r) = r\ln(a)$ (r a rational number).
   (c) Show that $\lim_{x\to\infty} \ln(x) = +\infty$ and $\lim_{x\to 0} \ln(x) = -\infty$ so that $\ln(x)$ has range $(-\infty, \infty)$.
   (d) Show that the inverse function $g = \ln^{-1}$ of ln is a strictly increasing differentiable function with domain $(-\infty,\infty)$ and we have $g'(y) = g(y)$.
   (e) Show that $g(y_1 + y_2) = g(y_1)g(y_2)$, $g(-y) = 1/g(y)$ and $g(ry) = g(y)^r$ for any rational number.

(f) Show that there is a unique number $e > 0$ such that $f(e) = 1$ and that $g(r) = e^r$ for any rational numbers.

6. Compute the following limits

   (a) $\lim_{x\to 0}(1 + 2x)^{1/x}$

   (b) $\lim_{x\to 0} \frac{e^{2x}-\cos(x)}{x}$

   (c) $\lim_{x\to 0} \left[\frac{1}{\sin(x)} - \frac{1}{x}\right]$

   (d) $\lim_{x\to 0} \cos(x)^{1/x^2}$

   (e) $\lim_{x\to 0} \frac{1-\cos(2x)-2x^2}{x^4}$.