Math 523H–Homework 1

1. Prove, by induction, that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

2. Show that $\sqrt{3}$ is not a rational number. *Hint:*: Maybe you can try to prove this by mimicking the proof for the irrationality of $\sqrt{2}$. Or you can use our theorem on algebraic number.

3. Is $(5+\sqrt{3})^{1/3}$ a rational number?

4. (a) Show that $|b| \le a$ if and only if $-a \le b \le a$.

   (b) Show that $\big||a| - |b|\big| \le |a-b|$ (this is called the *reverse triangle inequality* and is very useful).

   (c) Show that $|a-b| \le c$ if and only if $b - c \le a \le b + c$.

5. Compute the following limits. In this problem you should justify your claims using the definitions of limits: given an arbitrary ϵ provide a suitable $N = N(\epsilon)$.

   (a) $\lim_{n\to\infty} \frac{1}{n^{1/3}}$.

   (b) $\lim_{n\to\infty} \frac{7n-19}{3n+7}$.

   (c) $\lim_{n\to\infty} \frac{n+3}{n^2+1}$.

6. (a) Show that if $s_n \le a$ and $\lim_{n\to\infty} s_n = s$ then $s \le a$.

   (b) Show that if $s_n \le a$ for all but finitely many n and $\lim_{n\to\infty} s_n = s$ then $s \le a$.

   (c) If you assume $s_n < a$ in (a) or (b) (strict inequality) instead of $s_n \le a$ what can you conclude? Explain.

7. Assume that s_n are nonnegative numbers.

   (a) Show that if $\lim_{n\to\infty} s_n = 0$ then $\lim_{n\to\infty} \sqrt{s_n} = 0$.

   (b) Show that if $\lim_{n\to\infty} s_n = s$ then $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$.

8. Show that

   (a) $\lim_{n\to\infty}(\sqrt{n^2+1} - n) = 0$.

   (b) $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$.