## Lecture 3: Nash equilibrium

Nash equilibrium: The mathematician John Nash introduced the concept of an equilibrium for a game, and equilibrium is often called a Nash equilibrium. They provide a way to identify reasonable outcomes when an easy argument based on domination (like in the prisoner's dilemma, see lecture 2) is not available.

We formulate the concept of an equilibrium for a two player game with respective payoff matrices $P_{R}$ and $P_{C}$. We write $P_{R}\left(s, s^{\prime}\right)$ for the payoff for player $R$ when $R$ plays $s$ and $C$ plays $s$, this is simply the $\left(s, s^{\prime}\right)$ entry the matrix $P_{R}$.

Definition 1. A pair of strategies $\left(\hat{s}_{R}, \hat{s}_{C}\right)$ is an Nash equilbrium for a two player game if no player can improve his payoff by changing his strategy from his equilibrium strategy to another strategy provided his opponent keeps his equilibrium strategy.

In terms of the payoffs matrices this means that

$$
P_{R}\left(s_{R}, \hat{s}_{C}\right) \leq P\left(\hat{s}_{R}, \hat{s}_{C}\right) \quad \text { for all } s_{R},
$$

and

$$
P_{C}\left(\hat{s}_{R}, s_{C}\right) \leq P\left(\hat{s}_{R}, \hat{s}_{C}\right) \quad \text { for all } s_{c} .
$$

The idea at work in the definition of Nash equilibrium deserves a name:
Definition 2. A strategy $\hat{s}_{R}$ is a best-response to a strategy $s_{c}$ if

$$
P_{R}\left(s_{R}, s_{C}\right) \leq P\left(\hat{s}_{R}, s_{C}\right) \quad \text { for all } s_{R}
$$

i.e. $\hat{s}_{R}$ is such that

$$
\max _{s_{R}} P_{R}\left(s_{R}, s_{C}\right)=P\left(\hat{s}_{R}, s_{C}\right)
$$

We can now reformulate the idea of a Nash equilibrium as

The pair $\left(\hat{s}_{R}, \hat{s}_{C}\right)$ is a Nash equilibrium if and only if $\hat{s}_{R}$ is a best-response to $\hat{s}_{C}$ and $\hat{s}_{C}$ is a best-response to $\hat{s}_{R}$.

Finding Nash equilibrium: A very simple procedure allows to identify the Nash equilibrium by inspecting the payoff matrices $P_{R}$ and $P_{C}$. For $\left(\hat{s}_{R}, \hat{s}_{C}\right)$ we must have that $P_{R}\left(\hat{s}_{R}, \hat{s}_{C}\right)$ is a maximum of the entries on its column and $P_{C}\left(\hat{s}_{R}, \hat{s}_{C}\right)$ is a maximum of the entries on its row. This gives the easy algorithm

- Circle the maximum in each column of the matrix $P_{R}$.
- Circle the maximum in each row of the matrix $P_{C}$.
- If there is an entry $\left(s_{R}, S_{C}\right)$ which is circled in both $P_{r}$ and $P_{C}$ then $\left(s_{R}, S_{C}\right)$ is a Nash equilibrium.

Example: Consider the game with payoff matrices $P_{R}$ and $P_{C}$ given below. Circling the maxima on columns and rows we have

$$
P_{R}=\begin{gathered}
A \\
I I \\
0
\end{gathered}\left(\begin{array}{ccc}
A & B & C \\
0 & 0 & 0 \\
0
\end{array}\right) \quad P_{C}=\begin{gathered}
A \\
I \\
I I
\end{gathered}\left(\begin{array}{ccc} 
& B & C \\
0 & 4 & 3 \\
4 & 3 & 2
\end{array}\right)
$$

The entry corresponding to the pair of strategies $(I, B)$ is circled in both matrices $P_{R}$ and $P_{C}$ and thus is a Nash equilibrium.

Example: Nash equilibrium for the prisoner's dilemma: We have

$$
P_{R}=\begin{gathered}
C \\
N
\end{gathered}\left(\begin{array}{cc}
C & N \\
-6 & 0 \\
-8 & -1
\end{array}\right) \quad P_{C}=\begin{gathered}
C \\
N
\end{gathered}\left(\begin{array}{cc}
C & N \\
-6 & -8 \\
0 & -1
\end{array}\right)
$$

and thus the Nash equlibrium is $(C, C)$ as expected.
Example: Nash equilibrium in Battle of the sexes: We have

$$
P_{R}=\begin{gathered}
S \\
S \\
A
\end{gathered}\left(\begin{array}{cc}
S & A \\
1 & 3 \\
2 & 0
\end{array}\right) \quad P_{C}=\begin{gathered}
\\
S \\
A
\end{gathered}\left(\begin{array}{cc}
C & N \\
1 & 2 \\
3 & 0
\end{array}\right)
$$

and we have 2 Nash equilbria, namely $(S, A)$ and $(A, S)$.
Example: Nash equilibrium in the Matching Pennies game:

$$
P_{R}=\begin{gathered}
H \\
H \\
T
\end{gathered}\left(\begin{array}{cc}
H & T \\
1 & -1 \\
-1 & 1
\end{array}\right) \quad P_{C}=\begin{gathered}
H \\
H \\
T
\end{gathered}\left(\begin{array}{cc}
T \\
-1 & 1 \\
1 & -1
\end{array}\right)
$$

and we have no Nash equilbrium.

Pareto optimality. When applying game theory to social situation, think prisonner's dilemma or battle of the sexes, sometimes game theory yields to a outcome which seems not to be optimal from the point of view of social values. It sometimes seems like a "better" outcome which provides better payoffs to both players could occur. To quantify this we introduce the notion of Pareto optimal which is named after the economist Pareto.

Definition 3. A pair of strategies $\left(s_{R}, s_{C}\right)$ in a two-player game, is not Pareto optimal is there exists another choice of strategies $\left(s_{R}^{\prime}, s_{C}^{\prime}\right)$ such that both players are no worse off switching from $\left(s_{R}, s_{C}\right)$ to $\left(s_{R}^{\prime}, s_{C}^{\prime}\right)$ and at least one of the player is strictly better off $\left(s_{R}, s_{C}\right)$ to $\left(s_{R}^{\prime}, s_{C}^{\prime}\right)$. That is we have

$$
P_{R}\left(s_{R}^{\prime}, s_{C}^{\prime}\right) \geq P_{R}\left(s_{R}, s_{C}\right) \quad P_{C}\left(s_{R}^{\prime}, s_{C}^{\prime}\right)>P_{R}\left(s_{R}, s_{C}\right)
$$

or

$$
P_{R}\left(s_{R}^{\prime}, s_{C}^{\prime}\right)>P_{C}\left(s_{R}, s_{C}\right) \quad P_{R}\left(s_{R}^{\prime}, s_{C}^{\prime}\right) \geq P_{C}\left(s_{R}, s_{C}\right)
$$

For example the outcome predicted by game theory in the prisonner's dilemma is not Pareto optimal since by switching to not confess both players would be better off.

The fact that outcomes of a game are not always Pareto opimal should not be interpreted as a weakness of game theory. Sometimes being rational leads to socially destructive behavior and you can find plenty of such behavior in everyday life.

Symmetric game and social dilemma: Many of the examples in the previous lectures can be thought as describing social dilemma. These are games played by members of some group, who have the same incentives and interests although they will compete with each other.

We define a class of games which describe situation which are symmetric with respect to the players. In these games Robert and Collin are merely name (or label) assigned to players but the players are really interchangeable.

Definition 4. A two player game is symmetric if

- The set of strategies is the same for the two players.
- The players are interchangeable, i.e. the payoff for $R$ is $R$ plays $s$ and $C$ plays $s^{\prime}$ is the same as the payoff for $C$ if $C$ plays $s$ and $R$ plays $s^{\prime}$.

In terms of the matrices $P_{R}$ and $P_{C}$ we have

The game is symmetric if and only if $P_{R}$ and $P_{C}$ are transpose to each other $\left(P_{R}^{T}=\right.$ $P_{C}$ )

Nash equilibria for two-player, two strategies, symmetric games: For a symmetric game with strategies 1 and 2 the general payoff matrices have the form

$$
P_{R}=\begin{array}{cc}
1 & 2 \\
1 \\
2
\end{array}\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \quad P_{C}=\begin{array}{cc}
1 & 2 \\
1 \\
2
\end{array}\left(\begin{array}{ll}
a & c \\
b & d
\end{array}\right)
$$

Generically, if we exclude the cases where the some entries of the matrices coincide there are only three cases.

- Case (1.1) $a>c$ and $b>d$ : There is one Nash equilibrium (1,1). The equilibrium is Pareto efficient if and only if $a>d$.

$$
P_{R}=\begin{gathered}
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & b \\
c & d
\end{array}\right) \quad P_{C}=\begin{gathered}
1 \\
2 \\
\square
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & c \\
b
\end{array}\right)
$$

Case (1.2) $a<c$ and $b<d$ : There is one Nash equilibrium (2,2). The equilibrium is Pareto efficient if and only if $d>a$.

$$
P_{R}=\begin{gathered}
\\
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & b \\
c & d
\end{array}\right) \quad P_{C}=\begin{gathered}
1 \\
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & c \\
b & d
\end{array}\right)
$$

The case (1.2) is really the same as (1.1) (just interchange the names of strategies..)

- Case (2) $a>c$ and $b<d$ : There are 2 Nash equilibria $(1,1)$ and $(2,2)$ where the players pick the same strategy as their opponent. This type of game is called a coordination game.

$$
P_{R}=\begin{gathered}
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & b \\
c & (d)
\end{array}\right) \quad P_{C}=\begin{gathered}
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
\square & c \\
b & d
\end{array}\right)
$$

- Case (3) $a<c$ and $b>d$ : There are 2 Nash equilbria $(1,2)$ and $(2,1)$ where the players pick the opposite strategy as their opponent.

$$
P_{R}=\begin{gathered}
1 \\
1 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & b \\
c & d
\end{array}\right) \quad P_{C}=\begin{gathered}
1 \\
2 \\
2
\end{gathered}\left(\begin{array}{cc}
1 & 2 \\
a & C \\
b & d
\end{array}\right)
$$

## Exercises:

Exercise 1: Find all the Nash equilibria for the Chicken game. Are they Pareto optimal?
Exercise 2: Find all the Nash equilbria for the Rock-Scissor-Paper game. Are they Pareto optimal?

Exercise 3: Find all the Nash equilibria for the Snowdrift game (see Exercise 1 in Lecture 1). Are they Pareto optimal?

Exercise 4: Find all the Nash equilibria for the Ultimatum game (see Exercise 1 in Lecture 1). Are they Pareto optimal?

Exercise 5: A man has two sons Robert and Collin and when he dies the value of his estate is $\$ 100^{\prime} 000$. In his will it states that the two sons must each specify a sum of money $s_{R}$ and $s_{C}$ (in multiple of thousands) which they are willing to accept. If $s_{R}+s_{C} \leq 100000$ then they both get what they asked for and the remainder is sent to an animal shelter. If $s_{R}+s_{C}>100000$ then neither son receives any money and all the money goes to the animal shelter. Interpret this situation as a game where each son only cares about maximizing his profit and find all the Nash equilibria for this game.

Exercise 6: (Weakly dominated strategies) When we define domination we require that the payoff for the dominating strategy is strictly bigger than for the dominated one. Sometimes one uses the idea of weak domination: the strategy $s$ is weakly dominated by $s^{\prime}$ for the player $R$ if we have

$$
P_{R}\left(s, s_{C}\right) \leq P_{R}\left(s^{\prime}, s_{C}\right) \quad \text { for all } s_{C}
$$

and the equality is strict for at least one $s_{C}$.
One may argue that eliminating weakly dominated strategies is as good as eliminating dominated strategies to find the solution of the game.

1. Consider the game with payoff table

(a) Solve the game by eliminating iteratively the dominated strategies. Show that the solution you find depend on the order in which you eliminate the strategies.
(b) Compute all the Nash equilibria for the game.
2. Consider the game with payoff table

(a) Solve the game by eliminating iteratively (weakly) dominated strategies. Does it depend on the order with which you elimniated the dominated strategies?
(b) Compute all the Nash equilibria for the game.
